



Some properties of geodesic semi E-b-vex functions

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Abstract

This review study entitled “Some properties of geodesic semi E-b-vex functions” was conducted in Helmand Province, Afghanistan in 2020. The main aim of this paper is to present a new class of functions called geodesic semi E-b-vex functions and generalized geodesic semi E-b-vex functions and discusses some of their properties. The study was sought to define geodesic quasi-semi E-b-vex (GQSEB) functions and geodesic pseudo-semi E-b-vex (GPSEB) functions as generalizations of geodesic semi E-quasi convex functions and geodesic semi E-pseudo convex functions. A set $B \subseteq \mathbb{R}^n$ is called E-b-vex if there is such that $tbE(x_1) + (1-tb)E(x_2) \in B, \forall x_1, x_2 \in B, t \in [0,1]$ is definition given for Geodesic semi E-b-vex function and definition for Generalized geodesic semi E-b-vex functions: The mapping $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi semi E-b-vex on an E-b-vex set $B \subseteq \mathbb{R}^n$, if $h(tbE(x_1) + (1-tb)E(x_2)) \leq \max\{h(x_1), h(x_2)\}, \forall x_1, x_2 \in B, t \in [0,1]$. The paper is mostly based on secondary data and the definitions and properties that are used, can be found in many books on differential geometry.

Keywords: convexity, geodesic E-convexity, geodesic E-convex functions, nonlinear programming, riemannian manifolds

Introduction

Convex functions play an important role in optimization theory, convex analysis and fractal mathematics [1, 3, 8, 9, 10, 12, 15, 16]. In [21], Youness presented E-convexity of sets and functions. However, some results given by Youness [21] seem to be incorrect by Young [20]. Chen [2] extended E-convexity to semi E-convexity and discussed some of their properties. We refer to [17] for more results on E-convex functions or semi E-convex. A manifold is not a linear space and extensions of concepts and techniques from linear spaces to Riemannian manifolds are natural. There are many authors who studied generalized convex functions in Riemannian manifolds see [14, 18]. In 2012, Iqbal *et al.* [7] introduced and studied a new class of convex sets and functions which are called geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds. Recently, Iqbal *et al.* [6] introduced a new class of functions, namely geodesic semi E-convex functions. The main aim of this paper is to introduce a new class of functions, which are called geodesic semi E-b-vex (GSEB) functions, and to discuss some of their properties. We also define geodesic quasi-semi E-b-vex (GQSEB) functions and geodesic pseudo-semi E-b-vex (GPSEB) functions as generalizations of geodesic semi E-quasi convex functions and geodesic semi E-pseudo convex functions.

Preliminaries

In this section, we recall some definitions and properties, which will be used throughout the paper, These can be found in many books on differential geometry, such as [18].

Let N be a C^∞ n -dimensional Riemannian manifold, and $T_z N$ be the tangent space to N at z . Also, assume that $\mu_z(x_1, x_2)$ is a positive inner product on the tangent space $T_z N(x_1, x_2 \in T_z N)$, which is given for each point of N . then, a C^∞ map $\mu: Z \rightarrow \mu_z$, which assigns apposite inner product μ_z to $T_z N$ for each point z of N , is called a Riemannian metric.

The length of a piecewise C^1 curve $\eta: [a_1, a_2] \rightarrow N$ which is defined as follows:

$$L(\eta) = \int_{a_1}^{a_2} \|\dot{\eta}(x)\| dx.$$

We define that $d(z_1, z_2) = \inf\{L(\eta): \eta \text{ is a piecewise } C^1 \text{ curve joining } z_1 \text{ to } z_2\}$

for any points $z_1, z_2 \in N, \forall x, Y, X, Y \in N$ is a unique determined Riemannian connection, which is called Levi-Civita connection on every Riemannian manifolds.

Furthermore, a smooth path η is a geodesic if and only if its tangent vector η is a parallel vector field along the path η , i.e, η satisfies the equation. $\nabla_{\dot{\eta}(t)} \dot{\eta}(t) = 0$.

Every path η is joining $z_1, z_2 \in N$ where $L(\eta) = d(z_1, z_2)$ is a minimal geodesic.

Finally, assume that (N, μ) is a complete n-dimensional Riemannian manifold with Riemannian connection ∇ .

Let $x_1, x_2 \in N$ and $\eta: [0,1] \rightarrow N$ be a geodesic joining the points x_1 and x_2 , which means that $\eta_{x_1, x_2}(0) = x_2$ and $\eta_{x_1, x_2}(1) = x_1$.

Geodesic semi E-b-vex function

Firstly, let us give the following definitions

Definition3.1⁽¹³⁾. A set $B \subseteq \mathbb{R}^n$ is called E-b-vex if there is such that $tbE(x_1) + (1 - tb)E(x_2) \in B, \forall x_1, x_2 \in B, t \in [0,1]$

Definition3.2 ([13]). A function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is called E-b-vex on a set $B \subseteq \mathbb{R}^n$ if there is a map $E: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that B is a E-b-vex set and

$$h(tbE(x_1) + (1 - tb)E(x_2)) \leq tbh(E(x_1)) + (1 - tb)h(E(x_2)), \forall x_1, x_2 \in B, t \in [0,1]$$

We now replace the space \mathbb{R}^n by a Reimannian manifold N and introduce the concepts of geodesic E-b-vex sets and geodesic E-b-vex functions on a Riemannian manifold as follows:

Definition3.3. Assume $h: B \rightarrow \mathbb{R}, b: B \times B \times [0,1] \rightarrow \mathbb{R}_+$ are maps. A set $B \subseteq N$ is called geodesic E-b-vex set if there is a unique geodesic $\gamma^{E(x_1), E(x_2)}(tb)$ of length $d(x_1, x_2)$, which belongs to B, for all $x_1, x_2 \in B$ and $t \in [0,1]$.

Definition3.4. A function $h: B \rightarrow \mathbb{R}$ is called geodesic E-b-vex function on a set $B \subseteq N$ if there is a map $E: N \rightarrow N$, such that B is geodesic E-b-vex set and

$$h(\gamma^{E(x_1), E(x_2)}(tb)) \leq tbh(E(x_1)) + (1 - tb)h(E(x_2)), \forall x_1, x_2 \in B, t \in [0,1]$$

Lemma3.5. Let $E: N \rightarrow N$ such that E(B) is geodesic E-b-vex and E(B) geodesic E-b-vex on a set $B \subseteq N$ if h is geodesic E-b-vex on E(B).

Proosition3.6. Assume that $B \subseteq N$ is geodesic E-b-vex, then $E(B) \subseteq B$.

Proof. Since B is geodesic E-b-vex, then

$$\gamma^{E(x_1), E(x_2)}(tb) \in B, \forall x_1, x_2 \in B, t \in [0,1].$$

For $t=0$, we get $\gamma^{E(x_1), E(x_1)}(0) = E(x_2) \in B$, then $E(B) \subseteq B$. □

Proposition3.7. Assume that E(B) is geodesic E-b-vex and $E(B) \subseteq B$. Then B is geodesic E-b-vex.

Proof. Let $x_1, x_2 \in B$, then $E(x_1), E(x_2) \in E(B)$. $\gamma^{E(x_1), E(x_2)}(tb) \in E(B)$ Because E(B) is geodesic E-b-vex. Then $\gamma^{E(x_1), E(x_2)}(tb) \in B$ which implies that B is geodesic E-b-vex. □

Definition3.8. A function $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex on a set B if there is a map $E: N \rightarrow N$ such that B is a geodesic E-b-vex set and

$$h(\gamma^{E(x_1), E(x_2)}(tb)) \leq tbh(x_1) + (1 - tb)h(x_2), \forall x_1, x_2 \in B, t \in [0,1]$$

If the above inequality is strict for all $x_1, x_2 \in B, x_1 \neq x_2$ and $\forall t \in (0,1)$, then h is strictly geodesic semi E-b-vex.

Proposition3.9. Assume that a function $h: \rightarrow \mathbb{R}$ is a geodesic E—b-vex on a geodesic E-b-vex set B. Then h is geodesic semi E-b-vex on B if $h(E(x_1)) \leq h(x_1), \forall x_1 \in B$.

Proof. Let h be a geodesic semi E-b-vex on a geodesic E-b-vex set $B \subseteq N$, then $\gamma^{E(x_1), E(x_2)}(tb) \in B, \forall x_1, x_2 \in B$, we have

$$h(\gamma^{E(x_1), E(x_2)}(tb)) \leq tbh(x_1) + (1 - tb)h(x_2), \forall x_1, x_2 \in B, t \in [0,1].$$

If $tb=1$, then $h(E(x_1)) \leq h(x_1)$.

Conversely, let $h(E(x_1)) \leq h(x_1), \forall x_1 \in B$, then for any $x_1, x_2 \in B$ and $\forall t \in [0,1]$, we have

$$h(\gamma^{E(x_1), E(x_2)}(tb)) \leq tbh(E(x_1)) + (1 - tb)h(E(x_2)) \leq tbh(x_1) + (1 - tb)x(x_2).$$

Remark3.10. A geodesic E-b-vex function on geodesic E-b-vex set is not necessarily a geodesic semi E-b-vex function.

Example3.11. Let $h: \mathbb{R} \rightarrow \mathbb{R}$ such that $h(x) = -|x|$ and $E(x) = \alpha x, \alpha \in (0,1], \forall x \in \mathbb{R}$. We consider the geodesic \mathcal{Y} such that If $x_1, x_2 \geq 0$, then

$$yE(x_1), E(x_2)(tb) = \begin{cases} \frac{1}{\alpha} [E(x_2) + tb(E(x_1) - E(x_2))] & ; x_1 x_2 \geq 0, \\ \frac{1}{\alpha} [E(x_2) + tb(E(x_2) - E(x_1))] & ; x_1 x_2 < 0 \end{cases} = \begin{cases} x_2 + tb(x_1 - x_2) & ; x_1 x_2 \geq 0, \\ x_2 + tb(x_2 - x_1) & ; x_1 x_2 < 0. \end{cases}$$

If $x_1, x_2 \geq 0$, then

$$h(yE(x_1), E(x_2)(tb)) = h(x_2 + tb(x_1 - x_2)) = -|(1 - tb)x_2 + tbx_1| = -[(1 - tb)x_2 + tbx_1].$$

On the other hand

$$tbh(E(x_1)) + (1 - tb)h(E(x_2)) = tbh(\alpha x_1) + (1 - tb)h(\alpha x_2) = -\alpha [(1 - tb)x_2 + tbx_1].$$

Hence, $h(yE(x_1), E(x_2)(tb)) \leq tbh(E(x_1)) + (1 - tb)h(E(x_2)), \forall t \in [0,1], \alpha \in (0,1]$

Similarly, this inequality can be held good when $x_1, x_2 < 0$.

Now if $x_1 < 0$ and $x_2 > 0$, then

$$h(yE(x_1), E(x_2)(tb)) = h(x_2 + tb(x_2 - x_1)) = -|(1 + tb)x_2 - tbx_1| = -[(1 + tb)x_2 - tbx_1].$$

On other hand

$$tbh(E(x_1)) + (1 - tb)h(E(x_2)) = tbh(\alpha x_1) + (1 - tb)h(\alpha x_2) = -\alpha [(1 - tb)x_2 - tbx_1]$$

It follows that

$$h(yE(x_1), E(x_2)(tb)) \leq tbh(E(x_1)) + (1 - tb)h(E(x_2))$$

if

$$-[(1 + tb)x_2 - tbx_1] \leq -\alpha [(1 - tb)x_2 - tbx_1]$$

If

$$x_2(-1 - tb + \alpha(1 - tb)) + x_1 tb(1 - \alpha) \leq 0$$

Which is always true $\forall t \in [0,1], \alpha \in (0,1]$.

Similarly, this inequality can be also held good for $x_1 > 0, x_2 < 0$.

Thus h is geodesic E-b-vex function on R and since $h(E(1)) = h(\alpha) = -\alpha > f(1) = -1$ for $\alpha = \frac{1}{2}$, then from

Proposition3.9, h is not geodesic semi E-b-vex.

Remark3.12. from Proposition 3.9, it follows that geodesic E-b-vex function h on a geodesic E-b-vex set $B \subseteq \mathbb{N}$ with the property $h(E(x_1)) \leq h(x_1), \forall x_1 \in B$ is geodesic semi E-b-vex but the converse need not be true. In the example 3.11, if

$E(x_1) = \alpha x_1, \alpha > 1, \forall x_1 \in \mathbb{R}$, then the function h(x) is geodesic semi E-b-vex on geodesic E-b-vex set R while if $x_1 = 1, x_2 = 1$ and $tb = \frac{1}{2}$, then $h(yE(x_1), E(x_2)(tb)) = h(1) = -1$ while $tbh(E(x_1)) + (1 - tb)h(E(x_2)) = h(\alpha) = -\alpha, \alpha > 1$.

Hence, the function h is not geodesic E-b-vex on the geodesic E-b-vex set \mathbb{R} .

Theorem3.13. Let $h_1: B \rightarrow \mathbb{R}$ be a geodesic semi E-b-vex on geodesic E-b-vex-set $A \subseteq \mathbb{N}$. If $h_2: I \rightarrow \mathbb{R}$ is a non-decreasing E-b-vex function such that range $(h_1) \subset I$, then the composite function $h_1 \circ h_2$ is a geodesic semi E-b-vex on B.

Proof. Since h_1 is a geodesic semi E-b-vex on geodesic E-b-vex set B, then

$$h_1(yE(x_1), E(x_2)(tb)) \leq tbh_1(x_1) + (1 - tb)h_1(x_2), \forall x_1, x_2 \in B, t \in [0,1].$$

Now

$$\begin{aligned} h_1 \circ h_2(yE(x_1), E(x_2)(tb)) &\leq h_2 [tbh_1(x_1) + (1 - tb)h_1(x_2)] \\ &= tbh_2(h_1(x_1)) + (1 - tb)h_2(h_1(x_2)) = tbh_2 \circ h_1(x_1) + (1 - tb)h_2 \circ h_1(x_2). \end{aligned}$$

From the above it follows that $h_2 \circ h_1$ is geodesic semi E-b-vex on B. □

In addition, $h_2 \circ h_1$ is strictly geodesic semi E-b-vex function by considering h_2 to a strictly non- decreasing E-b-vex function.

Theorem 3.14. Let $B \subseteq N$ be a geodesic E-b-vex set and $h_i: B \rightarrow \mathbb{R}, i = 1, 2, \dots, z$ be geodesic semi E-b-vex function. Then $\sum_{i=1}^z \mu_i h_i, \forall \mu_i \in \mathbb{R}, \mu_i \geq 0, i = 1, 2, \dots, z$ is a geodesic semi E-b-vex on B.

Proof. Since each h_i is geodesic semi E-b-vex function on B, then

$$h_i(\gamma E(x_1), E(x_2)(tb)) \leq tb h_i(E(x_1)) + (1 - tb) h_i(E(x_2)).$$

It follows that

$$\mu_i h_i(\gamma E(x_1), E(x_2)(tb)) \leq tb \mu_i h_i(E(x_1)) + (1 - tb) \mu_i h_i(E(x_2))$$

Or

$$\sum_{i=1}^z \mu_i h_i(\gamma E(x_1), E(x_2)(tb)) \leq tb \sum_{i=1}^z \mu_i h_i(E(x_1)) + (1 - tb) \sum_{i=1}^z \mu_i h_i(E(x_2)).$$

Hence the result. □

Proposition 3.15. Assume that $\{h_i\}_{i \in I}$ is a family of real valued function defined on a geodesic E-b-vex set $B \subseteq N$ such that $\sup_{i \in I} h_i(x_1)$ exists in \mathbb{R} for all $x_1 \in B$. Assume that $h: B \rightarrow \mathbb{R}$ is a real function defined by $\sup_{i \in I} h_i(x_1), \forall x_1 \in B$. If $h_i: B \rightarrow \mathbb{R}, i \in I$ are geodesic semi E-b-vex function on B, then h is a geodesic semi E-b-vex function on B.

Proof. Since $h_i: B \rightarrow \mathbb{R}, \forall i \in I$ is a geodesic semi E-b-vex function on geodesic E-b-vex set B, then

$$h_i(\gamma E(x_1), E(x_2)(tb)) \leq tb h_i(x_1) + (1 - tb) h_i(x_2).$$

Then

$$\sup_{i \in I} h_i(\gamma E(x_1), E(x_2)(tb)) \leq \sup_{i \in I} [tb h_i(x_1) + (1 - tb) h_i(x_2)] = tb \sup_{i \in I} h_i(x_1) + (1 - tb) \sup_{i \in I} h_i(x_2).$$

This implies

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tb h(x_1) + (1 - tb) h(x_2).$$

Hence h is a geodesic semi E-b-vex function on B □

Proposition 3.16. Assume that $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$, then for any real number a the level set $M_a = \{x_1: x_1 \in B, h(x_1) \leq a\}$ is a geodesic E-b-vex set.

Proof. For any $x_1, x_2 \in M_a$ and $tb \in [0, 1]$, then $h(x_1) \leq a, h(x_2) \leq a$. Since h is a geodesic semi E-b-vex function, then

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tb h(x_1) + (1 - tb) h(x_2) = tba + (1 - tb)a = a.$$

Hence, M_a is a geodesic E-b-vex set.

4. Generalized geodesic semi E-b-vex functions

The concept of quasi E-b-vex function on \mathbb{R}^n was introduced by Mishra *et al.* [13] such as

Definition 4.1. The mapping $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is quasi semi E-b-vex on an E-b-vex set $B \subseteq \mathbb{R}^n$, if

$$h(tbE(x_1) + (1 - tb)E(x_2)) \leq \max\{h(x_1), h(x_2)\}, \forall x_1, x_2 \in B, t \in [0, 1]$$

We generalized the above concept and define geodesic quasi E-b-vex functions on Riemannian manifold and study some of their properties.

Definition 4.2. Assume that $B \subseteq N$ is a nonempty geodesic E-b-vex set. A function $h: B \rightarrow \mathbb{R}$ is called

1. Geodesic quasi semi E-b-vex if

$$h(\gamma E(x_1), E(x_2)(tb)) \leq \max\{h(x_1), h(x_2)\}, \forall x_1, x_2 \in B, t \in [0, 1].$$

2. Strictly geodesic semi quasi E-b-vex if

$$h(\gamma E(x_1), E(x_2)(tb)) < \max\{h(x_1), h(x_2)\}$$

$\forall x_1, x_2 \in B$ with $(x_1) \neq E(x_2)$ and $t \in (0,1)$.

Proposition 4.3. Assume that $\{h_i\}_{i \in I}$ is a family of real valued functions defined on a geodesic E-b-vex set $B \subseteq N$ such that $\sup_{i \in I} h_i(x_1)$ exists in \mathbb{R} for all $x_1 \in B$. Assume that $h: B \rightarrow \mathbb{R}$ is a real function defined by $\sup_{i \in I} h_i(x_1), \forall x_1 \in B$. If $h_i: B \rightarrow \mathbb{R}, i \in I$ are geodesic quasi semi E-b-vex functions on B, then h is a geodesic quasi semi E-b-vex function on B.

Proof. Since $h_i: B \rightarrow \mathbb{R}, \forall i \in I$ is a geodesic quasi semi E-b-vex function on a geodesic E-b-vex set B, then

$$\begin{aligned} h(\gamma E(x_1), E(x_2)(tb)) &= \sup_{i \in I} h_i(\gamma E(x_1), E(x_2)(tb)) \leq \sup_{i \in I} \max\{h_i(x_1), h_i(x_2)\} \\ &= \max\{\sup_{i \in I} h_i(x_1), \sup_{i \in I} h_i(x_2)\} = \max\{h(x_1), h(x_2)\}. \end{aligned}$$

Hence h is a geodesic quasi semi E-b-vex function on B.

Proposition 4.4. Assume that $B \subseteq N$ is a geodesic E-b-vex set. Then the function $h: B \rightarrow \mathbb{R}$ is geodesic quasi semi E-b-vex if and only if for any real number α the level set $M_\alpha = \{x_1: x_1 \in B, h(x_1) \leq \alpha\}$ is geodesic E-b-vex set.

Proof. Let h be a geodesic semi E-b-vex function on B. Thus, $h(\gamma E(x_1), E(x_2)(tb)) \leq \max\{h(x_1), h(x_2)\} \leq \alpha$. that implies to $\gamma E(x_1), E(x_2)(tb) \in M_\alpha$. thus, the set M_α is a geodesic E-b-vex.

Conversely, let $B \subseteq N$ be a geodesic E-b-vex set and M_α is a geodesic E-b-vex for each $\alpha \in \mathbb{R}$. Assume that $\alpha = \max\{h(x_1), h(x_2)\}$ for each $x_1, x_2 \in B$, then $x_1, x_2 \in M_\alpha$. Since M_α is a geodesic E-b-vex set, then $h(\gamma E(x_1), E(x_2)(tb)) \leq \alpha = \max\{h(x_1), h(x_2)\}$. Hence h is a geodesic quasi semi E-b-vex function on B.

Proposition 4.5. Assume that $g_i: N \rightarrow \mathbb{R}, i = 1, 2, \dots, k$ are geodesic quasi semi E-b-vex functions on N. Then the set $B = \{x_1 \in N: g_i(x_1) \leq 0, i = 1, 2, \dots, k\}$ is a geodesic E-b-vex set.

Proof. The proof follows from the proposition 4.4. \square

Proposition 4.6. Assume that $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$, then h is also a geodesic quasi E-b-vex function on B.

Proof. Since $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$, then

$$\begin{aligned} h(\gamma E(x_1), E(x_2)(tb)) &\leq tbh(x_1) + (1 - tb)h(x_2) \\ &\leq tb \max\{h(x_1), h(x_2)\} + (1 - tb) \max\{h(x_1), h(x_2)\} = \max\{h(x_1), h(x_2)\} \end{aligned}$$

Hence h is a geodesic quasi semi E-b-vex function on B.

In the following example, we can see that a geodesic quasi semi E-b-vex may not be geodesic semi E-b-vex.

Example 4.7. Assume that $h: B = [0,2] \rightarrow \mathbb{R}$ such that

$$h(x_1) = \begin{cases} 0 & ; 0 \leq x_1 \leq 1, \\ 1 & ; 1 < x_1 \leq 2 \end{cases} \quad E(x_1) = \begin{cases} 1 & ; 0 \leq x_1 \leq 1, \\ 2 & ; 1 < x_1 \leq 2 \end{cases}$$

and $b(x, y, t) = 1$.

We consider the geodesic \mathcal{Y} such that

$$\gamma E(x_1), E(x_2)(tb) = tbE(x_1) + (1 - tb)E(x_2), \forall x_1, x_2 \in [0,1].$$

It is clear that h is a geodesic quasi semi E-b-vex function but it is not a geodesic semi E-b-vex function on B set because, for $x_1 = 0, x_2 = 2, t = \frac{1}{2}$, we get

$$h(\gamma E(x_1), E(x_2)(tb)) = h\left(\frac{3}{2}\right) = 1 > tbh(x_1) + (1 - tb)h(x_2) = \frac{1}{2}.$$

Proposition 4.8. Assume that $h_1: B \rightarrow \mathbb{R}$ is a geodesic quasi semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$ and

$h_2: \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing function, then $h_2 \circ h_1$ is a geodesic quasi semi E-b-vex function on B.

Proof. Since $h_1: B \rightarrow \mathbb{R}$ is a geodesic quasi semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$ and $h_2: \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing function, then

$$(h_2 \circ h_1)(\gamma E(x_1), E(x_2)(tb)) = h_2(h_1(\gamma E(x_1), E(x_2)(tb))) \leq h_2(\max\{h_1(x), h_1(x_2)\}) \\ = \max\{h_2 \circ h_1(x_1), h_2 \circ h_1(x_2)\}$$

Hence $h_2 \circ h_1$ is a geodesic quasi semi E-b-vex function on B.

5. Epigraph and nonlinear programming

In this section, a characterization of geodesic E-b-vex function in terms of its epigraphs is discussed.

Definition 5.1. Assume that $B \subseteq N \times \mathbb{R}$ and $E: N \rightarrow N$, then the set B is called a geodesic E-b-vex set if

$$(\gamma E(x_1), E(x_2)(tb), tb a_1 + (1 - tb) a_2) \in B, \forall (x_1, a_1), (x_2, a_2) \in B, t \in [0,1]$$

Proposition 5.2. Assume that $\{B_i \subseteq N \times \mathbb{R}\}_{i \in I}$ is a family of geodesic e-b-vex sets. Then intersection $\bigcap_{i \in I} B_i$ is geodesic E-b-vex.

Proof. Let $(x_1, a_1), (x_2, a_2) \in \bigcap_{i \in I} B_i$ Then for each $i \in I, (x_1, a_1), (x_2, a_2) \in B_i$, we have

$$(\gamma E(x_1), E(x_2)(tb), tb a_1 + (1 - tb) a_2) \in B_i, \forall t \in [0,1].$$

Thus,

$$(\gamma E(x_1), E(x_2)(tb), tb a_1 + (1 - tb) a_2) \in \bigcap_{i \in I} B_i, \forall t \in [0,1].$$

Hence $\bigcap_{i \in I} B_i$ is a geodesic E-b-vex set.

Now, the epigraph $epi(h)$ of h is given by

$$epi(h) = \{(x_1, a_1): x_1 \in B \subseteq N, a_1 \in \mathbb{R}, h(x_1) \leq a_1\}.$$

A sufficient condition for h to be a geodesic semi E-b-vex function is given in the following theorem:

Theorem 5.3. Assume that $h: B \rightarrow \mathbb{R}$ is a real valued function on a geodesic E-b-vex set $B \subseteq N$. If $epi(h)$ is a geodesic E-b-vex set, then h is a geodesic E-b-vex function on B.

Proof. Let $x_1, x_2 \in B$ and $(x_1, h(x_1)), (x_2, h(x_2)) \in epi(h)$. Due to $epi(h)$ is geodesic E-b-vex set, then

$$(\gamma E(x_1), E(x_2)(tb), tb h(x_1) + (1 - tb) h(x_2)) \in epi(h),$$

So

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tb h(x_1) + (1 - tb) h(x_2).$$

Hence h is a geodesic E-b-vex function on B. □

Proposition 5.4. Assume that $\{h_i\}_{i \in I}$ is a family of real valued function which are bounded from above on a geodesic E-b-vex $B \subseteq N$ and let their epigraphs $epi(h_i)$ be geodesic E-b-vex set in $B \times \mathbb{R}$. Then, the function $h(x_1) = \sup_{i \in I} h_i(x_1)$ is a geodesic semi E-b-vex on B.

Proof. Since $epi(h_i) = \{(x_1, a_1): x_1 \in B, a_1 \in \mathbb{R}, h_i(x_1) \leq a_1\}$ are geodesic E-b-vex sets in $B \times \mathbb{R}$ Therefore, their intersection

$$\bigcap_{i \in I} epi(h_i) = \{(x_1, a_1): x_1 \in B, a_1 \in \mathbb{R}, h_i(x_1) \leq a_1, i \in I\} = \{(x_1, a_1): x_1 \in B, a_1 \in \mathbb{R}, h(x_1) \leq a_1\}$$

Is also geodesic E-b-vex set in $B \times \mathbb{R}$. Hence, by Theorem 5.3, h is geodesic semi E-b-vex function on B. □

Definition 5.5. Assume that $B \subseteq N \times \mathbb{R}, E: N \rightarrow N, I: \mathbb{R} \rightarrow \mathbb{R}$ and $b: B \times B \times [0,1] \rightarrow \mathbb{R}_+$, then a set B is called a geodesic $E \times I - b - vex$ set if $(\gamma E(x_1), E(x_2)(tb), tb a_1 + (1 - tb) a_2) \in B, \forall (x_1, a_1), (x_2, a_2) \in B, t \in [0,1]$

It is easy to show that $B \subseteq N$ is a geodesic E-b-vex set if $B \times \mathbb{R}$ is a geodesic $E \times I - b - vex$ set.

The following theorem gives a characterization of a geodesic E-b-vex function in terms of its $epi(h)$.

Theorem 5.6. Assume that $B \subseteq N$ is a geodesic E-b-vex set, then h is a geodesic semi e-b-vex function on B if $epi(h)$ is a geodesic $E \times I - v - vex$ on $B \times \mathbb{R}$.

Proof. Let be a geodesic semi E-b-vex function on B and let $(x_1, a_1), (x_2, a_2) \in epi(h), t \in [0,1]$, then

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tbh(x_1) + (1 - tb)h(x_2) \leq tba_1 + (1 - tb)a_2.$$

Thus,

$$(\gamma E(x_1), E(x_2)(tb), tba_1 + (1 - tb)a_2) \in \text{epi}(h)$$

Which implies that $\text{epi}(h)$ is geodesic $E \times I - b - \text{vex}$ on $B \times \mathbb{R}$.

Now, let $\text{epi}(h)$ be geodesic $E \times I - b - \text{vex}$ on $B \times \mathbb{R}$, and let $x_1, x_2 \in B, t \in [0, 1]$, then $(x_1, h(x_1)), (x_2, h(x_2)) \in \text{epi}(h)$. Due to $\text{epi}(h)$ being geodesic $E \times I - b - \text{vex}$ on $B \times \mathbb{R}$, then

$$(\gamma E(x_1), E(x_2)(tb), tba_1 + (1 - tb)a_2) \in \text{epi}(h)$$

That is

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tbh(x_1) + (1 - tb)h(x_2)$$

Which implies that h is a geodesic semi E-b-vex function on B . □

Definition 5.7. Let B be nonempty geodesic E-b-vex set. A function $h: B \rightarrow \mathbb{R}$ is called a geodesic semi E-b-vex on B , if there exist a strictly positive function $z: B \times B \rightarrow \mathbb{R}$ such that

$$h(x_1) < h(x_2) \rightarrow h(\gamma E(x_1), E(x_2)(tb)) \leq h(x_2) + tb(t - 1)z(x_1, x_2), \forall x_1, x_2 \in B, t \in (0, 1).$$

Proposition 5.8. Assume that $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$, then h is a geodesic pseudo semi E-b-vex on B .

Proof. Let $h(x_1) < h(x_2)$ and since h is a geodesic semi E-b-vex function on B , then $\forall x_1, x_2 \in B, t \in (0, 1)$

$$\begin{aligned} h(\gamma E(x_1), E(x_2)(tb)) &\leq tbh(x_1) + (1 - tb)h(x_2) < h(x_2) + tb(t - 1)(h(x_2) - h(x_1)) \\ &= h(x_2) + tb(t - 1)z(x_1, x_2) \end{aligned}$$

where $z(x_1, x_2) = h(x_2) - h(x_1) > 0$, then h is geodesic pseudo semi E-b-vex. □

Proposition 5.9. Assume that $h: B \rightarrow \mathbb{R}$ is a geodesic pseudo semi E-b-vex function on a geodesic E-b-vex set $B \subseteq N$, then h is a geodesic quasi semi E-b-vex on B .

Proof. Let $h(x_1) < h(x_2)$ and since h is a geodesic pseudo semi E-b-vex function on B , then $\forall x_1, x_2 \in B, t \in (0, 1)$, then

$$h(\gamma E(x_1), E(x_2)(tb)) \leq h(x_2) + tb(t - 1)z(x_1, x_2) < h(x_2) = \max\{h(x_1), h(x_2)\}$$

Hence h is a geodesic quasi semi E-b-vex on B .

Consider the following problem: $(P) \text{Min} h(x_1)$ such that $x_1 \in B = \{x_1 \in N: h_i(x_1) \leq 0, i = 1, 2, \dots, k\}$ are real valued function on a geodesic E-b-vex set B .

We also need to the following problem $(P_E) \text{Min}(h(E(x_1)))$ such that $x_1 \in B$.

Theorem 5.10. Assume that $B \subseteq N$ is a geodesic E-b-vex set and $h(E(x_1)) \leq h(x_1)$ for each $x_1 \in B$. If \bar{x}_1 is a solution of the problem (P_E) , then $E(\bar{x}_1)$ is a solution of the problem (P) .

Proof. Let $E(\bar{x}_1)$ be not a solution of problem (P) , then there exists $x_2 \in B$ such that $h(x_2) < h(E(\bar{x}_1))$, then $h(E(x_2)) \leq h(x_2) < h(E(\bar{x}_1))$, which contradicts the optimality of \bar{x}_1 of problem (P_E) .

Theorem 5.11. Assume that $h: B \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex on a geodesic E-b-vex set $B \subseteq N$ and \bar{x}_1 is a solution of the problem (P_E) , then $E(\bar{x}_1)$ is a solution of problem (P) .

Proof. The proof follows from the above theorem.

Theorem 5.12. Let $B \subseteq N$ be a geodesic E-b-vex set, $h: N \rightarrow \mathbb{R}$ be a geodesic E-b-vex function on B and $h(E(x_1)) \leq h(x_1), \forall x_1 \in B$ if $x^0 = E(z^0) \in E(B)$ is a local minimum of the problem (P) , then x^0 is global minimum of problem (P) on B .

Proof. Let $x^0 = E(z^0) \in E(B)$ be a nonglobal minimum of the problem (P) on B , then there is $x_2 \in B$ such that $h(x_2) < h(x^0) = h(E(z^0))$, since function $h: N \rightarrow \mathbb{R}$ is geodesic E-b-vex and $h(E(x_1)) \leq h(x_1), \forall x_1 \in B$, then

$$h(\gamma E(z^0), E(x_2)(tb)) \leq tbh(E(z^0)) + (1 - tb)h(E(x_2)) \leq tbh(x^0) + (1 - tb)h(x_2)$$

$$< tbh(x^0) + (1 - tb)h(x^0) = h(x^0)$$

for any small $t \in (0,1)$, which contradicts the local optimality of x^0 for problem (P). Hence x^0 is a global minimum of problem (P) on B.

Theorem 5.13. Assume that $h: N \rightarrow \mathbb{R}$ is a strictly geodesic semi E-b-vex on a geodesic E-b-vex set $B \subseteq N$, then the global optimal solution of problem (P) is unique.

Proof. Let $x_1 \neq x_2$ be two different global optimum solution of problem (P), then $h(x_1) = h(x_2)$. Since h is strictly geodesic semi E-b-vex on B, then

$$h(\gamma E(x_1), E(x_2)(tb)) < tbh(x_1) + (1 - tb)h(x_2) = h(x_1), \forall t \in (0, 1)$$

Which contradicts the optimality of x_1 of problem (P). Hence the global optimal solution of problem (P) is unique. \square

Theorem 5.14. Assume that $h: N \rightarrow \mathbb{R}$ is a geodesic quasi semi E-b-vex on a geodesic E-b-vex set $B \subseteq N$ and $\alpha = \min_{x_1 \in B} h(x_1)$. Then the set $G = \{x_1 \in B: h(x_1) = \alpha\}$ of optimal solutions of problem (P) is geodesic E-n-vex. If h is strictly geodesic quasi semi E-b-vex on B, then the set G is a singleton.

Proof. Let $x_1, x_2 \in G, t \in [0,1]$, then $x_1, x_2 \in B$ and $h(x_1) = \alpha = h(x_2)$. Since $h: B \rightarrow \mathbb{R}$ is geodesic quasi semi E-b-vex on B, then $h(\gamma E(x_1), E(x_2)(tb)) \leq \max\{h(x_1), h(x_2)\} = \alpha$ which implies that $\gamma E(x_1), E(x_2)(tb) \in G$ is geodesic E-b-vex.

Now, assume on the contrary that $x_1 \neq x_2 \in G$ and $t \in (0,1)$, then $\gamma E(x_1), E(x_2)(tb) \in B$. Since h is strictly geodesic quasi semi E-b-vex on B, then

$$h(\gamma E(x_1), E(x_2)(tb)) < \max\{h(x_1), h(x_2)\} = \alpha.$$

This contradicts that $\alpha = \min_{x_1 \in B} h(x_1)$ and hence the result.

Theorem 5.15. Assume that $h: N \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex on a geodesic E-b-vex set $B \subseteq N$, then the set of optimal solution of problem (P) is a geodesic E-b-vex.

Proof. Let x^* be optimal solution of problem (P) and let $\alpha = h(x^*)$. Assume that G is the set of optimal solutions for problem (P) as follows $G = \{x_1 \in B: h(x_1) \leq \alpha\}$, for any $x_1 \neq x_2 \in G$ and $t \in [0,1]$. Since $h: N \rightarrow \mathbb{R}$ is a geodesic semi E-b-vex function, then

$$h(\gamma E(x_1), E(x_2)(tb)) \leq tbh(x_1) + (1 - tb)h(x_2) \leq \alpha.$$

Thus, $\gamma E(x_1), E(x_2)(tb) \in G$, and it follows that G is geodesic E-b-vex set.

Theorem 5.16. Assume that $h: N \rightarrow \mathbb{R}$ and $g_i: N \rightarrow \mathbb{R}, i = 1, 2, \dots, k$ are quasi semi E-b-vex on N, then the set of optimal solution of problem (P) is a geodesic E-b-vex.

Proof. From proposition 4.5, it follows that B is geodesic E-b-vex set. Hence, by Theorem 5.14, the set $G = \{x_1 \in B: h(x_1) = \alpha\}$ of optimal solutions of problem (P) is geodesic E-b-vex.

Conflicts of interests

The authors declare that they have no conflicts of interests regarding publication of this article and they do not have direct financial relation that might lead to conflict of interest for any the author.

Acknowledgement

The authors are exceptionally grateful to the anonymous referees for their valuable suggestions and comment, which helped the authors to improve the work.

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