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On the ternary quadratic Diophantine equation $6(X^2 + Y^2)-11XY + 2(X + Y) + 4 = 27Z^2$

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Abstract
The ternary quadratic equation given by $6(X^2+Y^2)-11XY+2(X+Y)+4=27Z^2$ is considered and searched for its many different integer solutions. Four different choices of integer solutions to the above equation are presented. A few interesting relations between the solutions and special polygonal numbers are presented.

Keywords: ternary quadratic, integer solutions

1. Introduction
The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-16] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $6(X^2+Y^2)-11XY+2(X+Y)+4=27Z^2$ representing non-homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, few interesting relations among the solutions are presented.

2. Method of Analysis
The ternary quadratic diophantine equation to be solved for its non-zero distinct integral solution is

$$6(X^2+Y^2)-11XY+2(X+Y)+4=27Z^2 \tag{1}$$

Introducing the linear transformations $(u \neq v \neq 0)$

$$X=u+v, Y=u-v \tag{2}$$

In (1), it leads to

$$(u+2)^2+23v^2=27Z^2 \tag{3}$$

Different patterns of solutions of (1) are presented below:

2.1 Pattern-1
Write 27 as

$$27=(2+i\sqrt{23})(2-i\sqrt{23}) \tag{4}$$

Assume

$$Z=a^2+23b^2 \tag{5}$$

Where a, b are non-zero distinct integers.
Using (4) and (5) in (3), we get

$$(u+2)^2+23v^2=\left(2+i\sqrt{23}\right)\left(2-i\sqrt{23}\right)\left(a^2+23b^2\right)$$

Employing the method of factorization, we have

$$\left(u+2+i\sqrt{23}\,v\right)\left(u+2-i\sqrt{23}\,v\right)=\left(2+i\sqrt{23}\right)\left(2-i\sqrt{23}\right)\left(a+i\sqrt{23}\,b\right)^2\left(a-i\sqrt{23}\,b\right)^2 \quad (6)$$

Equating the positive and negative factors, we get

$$u+2+i\sqrt{23}\,v=\left(2+i\sqrt{23}\right)\left(a+i\sqrt{23}\,b\right)^2 \quad (6a)$$

$$u+2-i\sqrt{23}\,v=\left(2-i\sqrt{23}\right)\left(a-i\sqrt{23}\,b\right)^2 \quad (6b)$$

Equating real and imaginary parts either in (6a) or (6b) we get

$$u=u(a,b)=2a^2-46b^2-46ab-2$$
$$v=v(a,b)=a^2-23b^2+4ab$$

Substituting the values of u and v in (2) we get

$$X=X(a,b)=3a^2-69b^2-42ab-2 \quad (7)$$

$$Y=Y(a,b)=a^2-23b^2-50ab-2 \quad (8)$$

Thus (7), (8) and (5) represents non-zero distinct integral solutions of (1) in two parameters.

Properties

1. $3\Pr_a-X(a,1)-21\,GNO_a-113\equiv 0 \pmod 3$
2. $6\{Y(a,a)-X(a,a)\}$ is a nasty number
3. $Z(a,1)-X(a,1)+2\Pr_a-94\equiv 0 \pmod{11}$
4. $6\left(Z(a,a)-Y(a,a)-15\,t_{4,a}-2\right)$ is a nasty number
5. $Z(a,1)-t_{4,a}-23=0$
6. $X(a,1)-Y(a,1)-2\Pr_a+46\equiv 0 \pmod 3$

2.2 Pattern-2

Observe that (3) is written as

$$(u+2)^2+23v^2=23Z^2+4Z^2$$
$$(u+2)^2-4Z^2=23\left(Z^2-v^2\right) \quad (9)$$

Write (9) in the form of ratio as,

$$\frac{u+2+2Z}{Z+v}=\frac{23(Z-v)}{u+2-2Z}=\frac{\alpha}{\beta},\ \beta\neq 0 \quad (10)$$

Which is equivalent to the system of double equations

$$\Rightarrow u\beta + v\alpha + (8\beta - \alpha)Z = 0 \tag{11}$$

$$\Rightarrow u\alpha + 11v\beta - (8\alpha + 11\beta)Z = 0 \tag{12}$$

Applying the method of cross multiplication, we have

$$u = u(\alpha, \beta) = 2\alpha^2 - 46\beta^2 + 46\alpha\beta - 2$$
$$v = v(\alpha, \beta) = 23\beta^2 - \alpha^2 + 4\alpha\beta$$
$$Z = Z(\alpha, \beta) = \alpha^2 + 23\beta^2$$

Substituting the values of u and v in (2), we get

$$\left.\begin{aligned} X &= X(\alpha, \beta) = \alpha^2 - 23\beta^2 + 50\alpha\beta - 2 \\ Y &= Y(\alpha, \beta) = 3\alpha^2 - 69\beta^2 + 42\alpha\beta - 2 \\ Z &= Z(\alpha, \beta) = \alpha^2 + 23\beta^2 \end{aligned}\right\} \tag{13}$$

Thus (13) represents the non-zero distinct integer solutions to (1).

Note
Instead of (10), one may write (9) as

$$\frac{u + 2 + 2Z}{23(Z + v)} = \frac{Z - v}{u + 2 - 2Z} = \frac{\alpha}{\beta}, \beta \neq 0$$

Then the corresponding non-zero distinct integer solution to (1) are given by,

$$X = X(\alpha, \beta) = 23\alpha^2 - \beta^2 + 50\alpha\beta - 2$$
$$Y = Y(\alpha, \beta) = 69\alpha^2 - 3\beta^2 + 42\alpha\beta - 2$$
$$Z = Z(\alpha, \beta) = 23\alpha^2 + \beta^2$$

Properties

1. $Y(\alpha, \alpha + 2) + S_\alpha + 18\,\Pr_\alpha + GNO_\alpha + 278 \equiv 0 \pmod{2}$
2. $X(\alpha, 1) + Z(\alpha, 1) - 2\,\Pr_\alpha + 2 \equiv 0 \pmod{2}$
3. $Z(\alpha, \alpha - 1) - t_{46,\alpha} - 23 \equiv 0 \pmod{23}$
4. $Z(\alpha, \alpha + 1) - S_\alpha - 18\,\Pr_\alpha - 22 \equiv 0 \pmod{2}$

2.3 Pattern-3
Introduction of the linear transformations

$$Z = \alpha + 23\beta, \quad v = \alpha + 27\beta \tag{14}$$

In (3), leads to

$$(u + 2)^2 = 4\alpha^2 - 2484\beta^2 \tag{15}$$

Which is equivalent to the following systems of equations

Table 1: System of double equations

System	1	2	3	4	5	6
$2\alpha + u + 2$	β^2	$2\beta^2$	$54\beta^2$	$27\beta^2$	92β	9β
$2\alpha - u - 2$	2484	1242	46	92	27β	276β

Solving each of the above system of equations and using (2) and (14), the corresponding solutions to (1) are obtained. For simplicity, we present below the solutions.

Solutions for System-1

$X = 3k^2 + 54k - 623$

$Y = k^2 - 54k - 1865$

$Z = k^2 + 46k - 621$

Solutions for System-2

$X = 6k^2 + 60k - 284$

$Y = 2k^2 - 52k - 960$

$Z = 2k^2 + 48k + 334$

Solutions for System-3

$X = 162k^2 + 216k + 54$

$Y = 54k^2 - 50$

$Z = 54k^2 + 100k + 48$

Solutions for System-4

$X = 81k^2 + 54k - 25$

$Y = 27k^2 - 54k - 71$

$Z = 27k^2 + 46k + 23$

Solutions for System-5

$X = 357k - 2$

$Y = -97k - 2$

$Z = 211k$

Solutions for System-6

$X = -141k - 2$

$Y = -927k - 2$

$Z = 377k$

2.4 Pattern-4

One may write (3) as

$$(u + 2)^2 + 23v^2 = 27Z^2 * 1 \qquad (16)$$

Write 1 as

$$1=\frac{(11+i\sqrt{23})(11-i\sqrt{23})}{12^2} \qquad (17)$$

Assume

$$Z=a^2+23b^2 \qquad (5)$$

Where a, b are non-zero distinct integers.
Using (4), (5) and (17) in (16), we get

$$(u+2)^2+23v^2=(2+i\sqrt{23})(2-i\sqrt{23})(a^2+23b^2)\frac{(11+i\sqrt{23})(11+i\sqrt{23})}{12^2}$$

Employing the method of factorization, we have

$$(u+2+i\sqrt{23}v)(u+2-i\sqrt{23}v)=(2+i\sqrt{23})(2-i\sqrt{23})\frac{(11+i\sqrt{23})}{12}\frac{(11-i\sqrt{23})}{12}(a+i\sqrt{23}b)^2(a-i\sqrt{23}b)^2 \qquad (18)$$

Equating the positive and negative factors, we get

$$u+2+i\sqrt{23}v=(2+i\sqrt{23})(a+i\sqrt{23}b)^2\frac{(11+i\sqrt{23})}{12} \qquad (18a)$$

$$u+2-i\sqrt{23}v=(2-i\sqrt{23})(a-i\sqrt{23}b)^2\frac{(11-i\sqrt{23})}{12} \qquad (18b)$$

Equating real and imaginary parts either in (18a) or (18b) we get

$$u=u(a,b)=\frac{1}{12}\{-a^2+23b^2-598ab-24\}$$

$$v=v(a,b)=\frac{1}{12}\{13a^2-299b^2-2ab\}$$

Replacing a by 12A and b by 12B, we get

$$u=u(A,B)=-12A^2+276B^2-7176AB-2$$

$$v=v(A,B)=156A^2-3588B^2-24AB$$

Substituting the values of u and v in (2) we get

$$X=X(A,B)=144A^2-3312B^2-7200AB-2 \qquad (19)$$

$$Y=Y(A,B)=-168A^2+3864B^2-7152AB-2 \qquad (20)$$

$$Z=Z(A,B)=144A^2+3312B^2 \qquad (21)$$

Thus (19), (20) and (21) represents non-zero distinct integral solutions of (1) in two parameters.

Properties

1. $6\{Z(A,1)-3312\}$ is a nasty number
2. $X(A,1)-24S_A+3338\equiv 0(\mathrm{mod}\ 2)$
3. $Z(1,B)-3312t_{4,B}-144=0$
4. $-Y(A,1)-28S_A+3390\equiv 0(\mathrm{mod}\ 2)$

3. Conclusion

In this paper, we have presented infinitely many non-zero distinct integer solutions to the ternary quadratic equation $6(X^2+Y^2)-11XY+2(X+Y)+4=27Z^2$ representing a homogeneous cone. Diophantine equations are rich in variety.

To conclude, one may search for other forms of three dimensional surfaces, namely, non-homogeneous cone, paraboloid, ellipsoid, hyperboloid, hyperbolic paraboloid and so on for finding integral points on them and corresponding properties

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