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Teaching epsilon-delta definition of limit in class

Patanjali Sharma, PK Chaurasia

Department of Education in Science and Mathematics, Regional Institute of Education (National Council of Educational Research and Training), Ajmer, Rajasthan, India

Abstract

The process of limit is the heart of Calculus. Most of the students in the class are interested in the techniques of finding the limit, rather than getting concept of limit. This happens, due to the usual practice of mathematics teachers to introduce the concept and techniques of derivative first and, limit later in the class. Almost each Textbook of Calculus and Analysis first introduces the concept of limit and then derivative, but in the classroom often the process is in reverse order. Our experience among U.G. students of Calculus, many students face difficulty in understanding the Epsilon-Delta definition of limit, however they have been using the limit process in their courses of study since two-three years. The aim of this paper is to show, how one can introduce Epsilon-Delta definition of limit in class before introducing derivative. We believe that after learning with steps or methods presented in the paper students will definitely get inside the concept of limit and will be able to explain it.

Keywords: Limit, Epsilon-Delta Definition, Calculus

1. Introduction

The process of limit is the heart of Calculus. Most of the students in the class are interested in the techniques of finding the limit, rather than getting concept of limit. This happens, due to the usual practice of mathematics teachers to introduce the concept and techniques of derivative first, and limit later, in the class. Almost each Textbook of Calculus and Analysis first introduces the concept of limit and then derivative, but in the classroom often the process is in reverse order.

The concept of limit and continuity was initially given by Sir Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716). However, mathematicians had an imperfect understanding of these fundamental ideas even as late as the last century. Definitions of the limit given by Augustin-Louis Cauchy (1789-1857) and others referred to variables approaching indefinitely a fixed value and frequently made use of infinitesimals, quantities that become infinitely small but not zero.

Definition 1

Let $f(x)$ be defined on an open interval about a point 'a', except possible at 'a' itself. Then L is the limit of $f(x)$ at 'a', and is written as $\lim_{x\to a} f(x) = L$, if for all $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$$

The now accepted Epsilon-Delta ($\epsilon - \delta$) definition of limit given above was formulated by German mathematician Karl Weierstrass (1815-1897) in the middle of the nineteenth century as part of his attempt to put mathematical analysis on a sound logical foundation.

2. Idea of limit

In order to give the idea of limit we propose the following example of a function with the answers given during the classroom situation:

T: What is the value of $\mathrm{f(x)} = \frac{\mathrm{x}^2-9}{\mathrm{x}-3}$ at $\mathrm{x} = 3$?

S: Not defined, Cannot determine?

T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 3.1$?

S: After doing some calculations: 6.1

T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 3.01$?

S: After doing some calculations: 6.01

T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 3.001$?

S: After doing some calculations: 6.001

T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 3.0001$?

S: After doing some calculations: 6.0001
T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 2.9$?
S: After doing some calculations: 5.9
T: What is the value of $f(x) = \frac{x^2-9}{x-3}$ at $= 2.99$?
S: After doing some calculations: 5.99
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S: After doing some calculations: 5.999
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S: After doing some calculations: 5.9999

The above discussion may be summarize in the following tables:

Table 1

x	$f(x)$
3.1	$f(3.1) = \frac{3.1^2 - 9}{3.1 - 3} = 6.1$
3.01	$f(3.01) = \frac{3.01^2 - 9}{3.01 - 3} = 6.01$
3.001	$f(3.001) = \frac{3.001^2 - 9}{3.001 - 3} = 6.001$
3.0001	$f(3.0001) = \frac{3.0001^2 - 9}{3.0001 - 3} = 6.0001$

Table 2

x	$f(x)$
2.9	$f(2.9) = \frac{2.9^2 - 9}{2.9 - 3} = 5.9$
2.99	$f(2.99) = \frac{2.99^2 - 9}{2.99 - 3} = 5.99$
2.999	$f(2.999) = \frac{2.999^2 - 9}{2.999 - 3} = 5.999$
2.9999	$f(2.9999) = \frac{2.9999^2 - 9}{2.9999 - 3} = 5.9999$

T: When x approaches 3, then $f(x)$ goes near to which number?
S: 6.
T: We may also say that, $\lim_{x\to 3} f(x) = 6$ or may write

$$\lim_{x\to 3} f(x) = 6$$

or

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$$

3. Observations

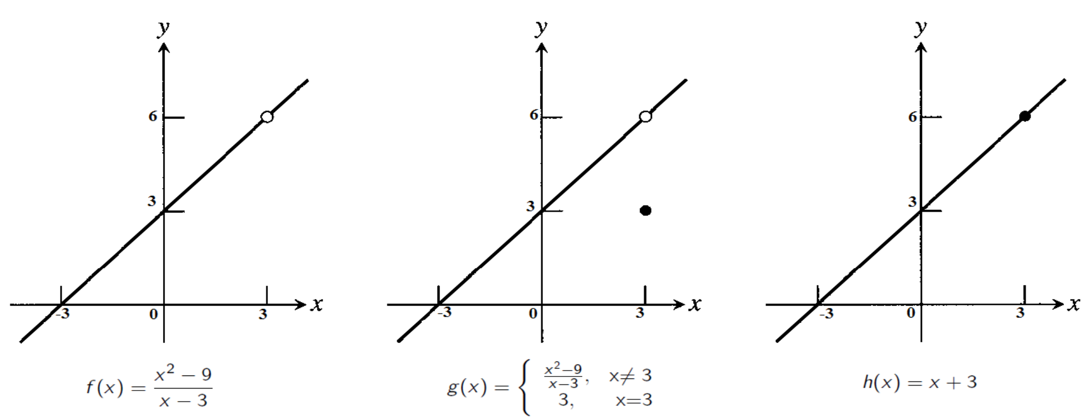

Fig 1: Three different possible situations for the image of a point

The above figure was shown to the students as three different possible situations for the image of a point

Formal Definition of limit

Let $f(x)$ be defined on an open interval about a point 'a', except possible at 'a' itself. We say that $f(x)$ approaches the limit L as x approaches 'a', and write

$$\lim_{x \to a} f(x) = L$$

If, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$$0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

T: Try to rephrase the definition of limit in your own words ?
S1: What is ϵ ?
S2: What is δ ?
S3: How clumsy these expressions are ?
S4: What is mean by $0 < |x - a| < \delta$?

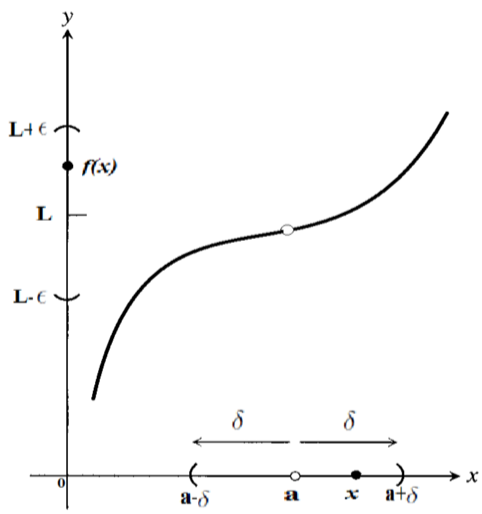

Fig 2: Visualizing the possible positions of ϵ and δ.

Further, we explain the above Fig. 2 in connection to the formal definition of the limit. This figure is quite helpful in explaining the definition of limit. Then we introduce the following questions to the students and if they feel difficulty in solving them we provide them solutions as well as steps involved in the process.

Problem 1

For input x, in order to keep output $f(x) = 2x - 1$, restricted between 2 units of 7. How close $x = 4$ must we hold ?
i.e. For what values of x, the function $[f(x) - 7]$ is restricted between 2 units of 7 ?

We have

$$|f(x) - 7| < 2$$

$\Rightarrow |(2x - 1) - 7| < 2$
$\Rightarrow |2x - 8| < 2$
$\Rightarrow -2 < 2x - 8 < 2$
$\Rightarrow 6 < 2x < 10$
$\Rightarrow 3 < x < 5$
$\Rightarrow -1 < x - 4 < 1$

Thus keeping x within 1 unit of $x = 4$ will keep $f(x)$ within 2 units of 7.

Problem 2

If $f(x) = x + 1, L = 5, a = 4, \epsilon = 0.01$. Then find a value for $\delta > 0$ such that for all x satisfying $0 < |x - a| < \delta$ the inequality $|f(x) - L| < \epsilon$

Step 1

Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (u, v) about a on which the inequality holds for all $x \neq a$.

Step 2
Find a value of $\delta > 0$ that places the open interval $(a - \delta, a + \delta)$ centred at a inside the interval (u, v), i.e., minimum of $(a - u, v - a)$. The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq a$ in this δ-interval.

Problem 3
Prove that: $\lim_{x \to 1} f(x) = 1$. If,

$$f(x) = \begin{cases} x^2, x \neq 1 \\ -2, x = 1 \end{cases}$$

Step 1
Solve the inequality $|f(x) - L| < \epsilon$ to find an open interval (u, v) about a on which the inequality holds for all $x \neq a$.

Step 2
Find a value of $\delta > 0$ that places the open interval $(a - \delta, a + \delta)$ centered at a inside the interval (u, v), i.e., minimum of $(a - u, v - a)$. The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq a$ in this δ-interval.

4. Conclusions
In this paper we share our experience of teaching Epsilon-Delta definition of limit in class and how one can teach limit, before introducing derivative in the class. We believe that our experience is helpful to the students or teachers who wish to introduce limit before derivative. For more details one can consult the excellent texts (see [1, 2, 3]).

5. References

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