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For a Nilpotent p-group and its center

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Abstract
Let G be a group. We give some definitions and results. We find the condition that order of the center of a nilpotent p-group G is not equal to p, where p is a prime.

Keywords: center, nilpotentency class, central series

1. Introduction

Let G be a group Z(G) denote the center of the group and G' denote the commutator subgroup of G. The number of elements of a group is called its order, the order of G is denoted by |G|. The order of an element a in a group G is the smallest positive integer n such that $a^n = e$, where e is the identity of G. If no such integer exists, a has infinite order. The order of an element a is denoted by o(a). Z (G), the center of a group G is the subset of those elements in G which commutes with every element in G. We can write $Z(G) = \{a \varepsilon G \mid ax = xa \ \forall x \varepsilon G\}$. The centralizer of a in G denoted by C(a). It is the set of elements in G that commute with a. We can write $C(a) = \{ x \varepsilon G \mid ax = xa \}$. A group G is called cyclic. If there is an element g in G such that $G = \{ g^n \mid n \varepsilon Z\}$. The element g is called a generator of G. We call G is generated by g and write G = <g>. A subgroup H of a group G is called a normal subgroup of G. If Ha = aH for all a in G [1].

2. Definitions

A sequence of subgroups

$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_m = \{e\}$ of a group G is called a subnormal series of G, If G_{i+1} is a normal subgroup of $G_i \ \forall$ i = 0, 1, ... m-1.

Where G_i/G_{i+1} are called factor groups of subnormal series. If each Gi is a normal subgroup of G, then the series is said to be a normal series of G [3].

An irredundant subnormal series

$G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_m = \{e\}$ of a group G is said to be a composition series of G, If each of its factor groups G_i/G_{i+1} is a simple group. We can also say that G_{i+1} is a simple group. We can also say that G_{i+1} is a maximal normal subgroup of G_i [3].

Two subnormal series

$G = G_o \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_m = \{e\}$

And $G = G'_o \supseteq G'_1 \supseteq G'_2 \supseteq \dots \supseteq G'_m = \{e\}$

Of a group G are said to be equivalent. If there exists a one-one correspondence between factor groups of both series such that the corresponding factor groups are isomorphic [3].

[2] A finite collection of normal subgroups H_i of a group G is a normal series for G if

$1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq H_r = G$. This normal series is a central series if $H_i/H_{i-1} \subseteq Z(G/H_{i-1})$ for $1 \leq i \leq r$. A group G is nilpotent if it has a central series. Subgroups and factor groups of nilpotent groups are nilpotent.

Given any group G, we define a central series as follows. Let $Z_0 = 1$ and $Z_1 = Z(G)$. The second center Z_2 is defined to be the unique subgroup such that $Z_2/Z_1 = Z(G/Z_1)$. We continue like this, inductively defining Z_n for n > 0 so that $Z_n/Z_{n-1} = Z(G/Z_{n-1})$.

The chain of normal subgroups.

$1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$

Constructed in this way is called upper central series of G. The upper central series may not actually be a central series for G because it may happen that $Z_i < G$ for all i. If $Z_r = G$ for some integer r, then $\{Z_i \mid 0 \leq i \leq r\}$ is a true central series and G is nilpotent.

If G is an arbitrary nilpotent group, then G is a term of its upper central series. As $G = Z_r$ for some integer $r \geq 0$, and the smallest integer r for which this happens is called the nilpotence class of G. Non trivial abelian groups have nilpotence class 1, and for non-abelian groups of nilpotency class 2 have quotient group G/Z(G) abelian.

2.1 Lemma

Let G be finite. Then the following are equivalent [2].

1. G is nilpotent
2. Every nontrivial homomorphic image of G has a nontrivial center.
3. G appears as a member of its upper central series.

2.2 Theorem

Let G be a (not necessarily finite) nilpotent group with central series [2].

$1 = 1 = H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq H_r = G$,

and let $1 = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$

be the upper central series of G. Then $H_i \subseteq Z_i$ for $0 \leq i \leq r$, and in particular, $Z_r = G$.

Theorem 2.3: Let H is a subgroup of G, where G is a nilpotent group. Then $N_G(H) > H$ [2].

Theorem 2.4: For any group G, Inn (G) is a normal subgroup of Aut (G). Further Inn (G) ≅ G/Z(G), where Z(G) denotes the centre of G.

3. Results

Theorem 3.1 Let G be a group, g ε G. If g has infinite order then all distinct powers of g are distinct elements. If g has finite order n, then $g^i = g^j$ if and only if n divides i – j [1].
Theorem 3.2 If $g^t = e$ then o(g) divides t [1].
Theorem 3.3 Let G = <g> be a cyclic group of order n. Then G = <g^k > If and only If (k, n) = 1 [1]
Theorem 3.4 If G is a finite group and H is a subgroup of G then order of H divides order of G [1].
Theorem 3.5 Let a ε G, (a) divides |G|, |G| is order of G [1].
Theorem 3.6 Groups of prime order are cyclic [1].
Theorem 3.7 Let a ε G then $a^{|G|} = \{e\}$ [1]
Theorem 3.8 Let G be a group and let Z(G) be the center of G. If G/Z(G) is cyclic, then G is abelian [1].
Theorem 3.9 For any group G, G/Z(G) is isomorphic to Inn (G) [1].
Theorem 3.10 Let G be a finite abelian group and let p be a prime that divides the order of G. Then G has an element of order p [1].
Theorem 3.11 If a prime p divides the order of a finite group G then G has at least one element of order p [3].
Theorem 3.12 Every Abelian group G of order n, is an internal direct product $G = G_1 \times G_2 \times \ldots\ldots \times G_m$ [3]
Where G_i are cyclic subgroups of order n_i such that n_{i+1} divides n_i and the integers n_i are unique. Also we have $n = n_1 n_2 \ldots\ldots n_m$
Theorem 3.13 Let G be a group and G' be its commutator subgroup, then [3]
(1) G' is a normal subgroup of G.
(2) Let H be a normal subgroup of G, than G/H is an abelian group If and only If $G' \subseteq H$
Theorem 3.14 Let H and K be two subgroups of G, H_0 and K_0 be normal subgroups of H and K. Then [3].

$$\frac{K_0(K \cap H)}{K_0(K \cap H_0)} \cong \frac{H_0(H \cap K)}{H_0(H \cap K_0)}$$

4. Let $|G| = p^n$, with nilpotency class c and $n \geq 2c$ then $|Z(G)| \neq p$, where p is a prime number.
Example (1) If $|G| = p^6$ and c(G) = 3,
Where c(G) is nilpotency class of G.
$Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq Z_3 = G$
$|Z_2| = p^5$ or p^4, because G/Z_2 is abelian
If $|Z_2| = p^5$ then $|Z_1| = p^4$ or p^3 since Z_2/Z_1 is abelian
If $|Z_2| = p^4$ then $|Z_1| = p^3$ or p^2

Where $Z_1 = Z(G)$, Therfore $|Z(G)| \neq p$.
Example (2) Let $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq Z_4 = G$
be the upper central series of G such that $|G| = p^8$ with c(G) = 4
Here $|Z_3| = p^7$ or p^6,
If $|Z_3| = p^7$ then $|Z_2| = p^6$ or p^5
If $|Z_3| = p^6$ then $|Z_2| = p^5$ or p^4
Let $|Z_2| = p^4$ then $|Z_1| = p^3$ or p^2
Therefore we have $|Z(G)| \neq p$.

Example (3) If we take the case when
c(G) = c and n < 2 c
Take $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq Z_3 = G$

Where $|G| = p^5$
Then $|Z_2| = p^4$ or p^3
from here we have $|Z_1|$ may be p, p^2 or p^3
Example (4) If $o(G) = p^7$ and c(G) = 3
Let $Z_o \subseteq Z_1 \subseteq Z_2 \subseteq Z_3 = G$ be the upper central series of G.
Since $o(G) = p^7$, we have $o(Z_2) = p^6$ or p^5
If $o(Z_2) = p^6$, then $0(Z_1) = p^4$ or p^5 so that Z_2/Z_1 is abelian. We can say that $o(Z_1) \neq p$.
This result is open for discussion

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