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On certain subclasses of p-Valent analytic functions and (m,q)-starlike with respect to certain points associated with generalized basic hypergeometric function

[1] Lokesh Jasoria, [2] SK Bissu
[1] Government Girl's College, Chittorgarh, Rajasthanp, India
[2] Government College of Ajmer, Ajmer, Rajasthan, India

Abstract
In this Paper we have introduced and study subclasses of univalent functions with negative coefficient by using Dziok-Srivastava operator defined in punchered open unit disk. Here we have studied coefficient estimates, Distortion Theorems, Extreme Points. These results include many results as particular cases.

Keywords: univalent functions, coefficient estimates, distortion bounds, extreme points

Introduction

Let C_p denote the class of functions f(z) of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} \alpha_{k+p} z^{k+p} \tag{1}$$

which are analytic and p-valent in the punchered open unit disk U = {z: $|z| < 1$} and normalized by f' (0) = f (0) +1 = 1. Let f $\in$ C_p given by (1) and g $\in$ C_p given by

$$g(z) = z^p + \sum_{k=1}^{\infty} \beta_{k+p} z^{k+p} \tag{2}$$

We define the convolution product (or Hadamard) of f and g by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} \alpha_{k+p} \beta_{k+p} z^{k+p} = (g*f)(z); \qquad (z \in U) \tag{3}$$

For +ve real parameters a_1, a_2, -----, a_m and b_1, b_2, ------, b_q.
$\left(b_j \in \mathbb{C}/\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0, -1, -2, \dots \dots \dots; j = 1,2, \dots \dots, q\right)$, the generalized hyper geometric function

${}_mF_q$ (a_1, a_2, -----------, a_m, b_1, b_2, -------------, b_q ; z) is defined by

$${}_mF_q\ (a_1, a_2, \dots \dots \dots \dots \dots a_m; b_1, b_2, \dots \dots \dots \dots b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots \dots \dots (a_m)_n}{(b_1)_n \dots \dots \dots (b_q)_n} \cdot \frac{1}{n!} z^n$$

$(m \le q+1; m, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2, \dots\}; z \in U)$ Where $(\emptyset)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$(\emptyset)_n = \frac{\Gamma(\phi + n)}{\Gamma(\phi)} = \begin{cases} 1 & , n = 0 \\ \phi(\phi+1) \dots \dots (\phi + n - 1) & , n \in \mathbb{N} \end{cases}$$

For the function

$$h(a_1, a_2, ---, a_m, b_1, b_2, ----, b_q; z) = z\ {}_qF_s(a_1, a_2, ----, a_m, b_1, b_2, ------, b_q; z),$$

the Dziok-Srivastava linear operator (see [1]).

$$H_{m,q}(a_1, a_2, \text{---------}, a_m, b_1, b_2, \text{--------}, b_q) : C_p \to C_p$$

is defined by the Hadamard product as follows:

$$H_{m,q}(a_1, a_2, \text{---------}, a_m, b_1, b_2, \text{--------}, b_q) f(z) = h(a1, a_2, \text{---------}, a_m, b_1, b_2, \text{--------}, b_q; z) * f(z)$$

$$= z^p + \sum_{n=1}^{\infty} \Gamma_n(a_1) a_{n+p} z^{n+p} \qquad (z \in U) \tag{4}$$

Where

$$\Gamma_n(a_1) = \frac{(a_1)_{n-1} \cdots\cdots (a_m)_{n-1}}{(b_1)_{n-1} \cdots\cdots (b_q)_{n-1}} \cdot \frac{1}{(n-1)!} \tag{5}$$

For brevity, we write

$$H_{m,q}(a_1, a_2, \text{----}, a_m, b_1, b_2, \text{----}, b_q; z) f(z) = H_{m,q}(a_1) f(z)$$

Definition 1. Let the function f (z) be defined by (1) then $f(z) \in S_{m,q}(z)$ if and only if

$$Re \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z)} > 0. \tag{6}$$

Definition 2. Let the function f (z) be defined by (1) then $f(z) \in S^*_{m,q}(z)$ be subclass of C_p if and only if

$$Re \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - H_{m,q}(a_1) f(-z)} > 0, \tag{7}$$

these classes of functions are called starlike with respect to symmetric points in U.
Let T_p denotes the subclasses of C_p consisting of functions of the form :

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (a_{k+p} \geq 0.) \tag{8}$$

Definition 3. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - H_{m,q}(a_1) f(-z)} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - H_{m,q}(a_1) f(-z)} + p \right| \tag{9}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to symmetric points by $S^*_{s,m,q}\, T_p(z)$.

Definition 4. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to conjugate points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) + \overline{H_{m,q}(a_1) f(\bar{z})}} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - \overline{H_{m,q}(a_1) f(\bar{z})}} + p \right| \tag{10}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to conjugate points by $S^*_{c,m,q}\, T_p(z)$.

Definition 5. Let the function f(z) be defined by (8). Then f(z) is said to be m-q starlike with respect to symmetric conjugate points if it satisfies the following condition:

$$\left| \frac{z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) - \overline{H_{m,q}(a_1) f(\overline{-z})}} - p \right| < \beta \left| \frac{\alpha\, z\left(H_{m,q}(a_1) f(z)\right)'}{H_{m,q}(a_1) f(z) + \overline{H_{m,q}(a_1) f(\overline{-z})}} + p \right| \tag{11}$$

For some $0 < \beta \leq 1, 0 \leq \alpha \leq 1$ *and* $z \in U$. We denote the class m-q-starlike with respect to symmetric conjugate points by $S^*_{sc,m,q}\, T_p(z)$.

2. Coefficient Estimates

Theorem 1. Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z) \neq 0$ for $z \neq 0$. Then f(z) is in the class $S^*_{s,m,q}\, T_p(z)$ if and only if

$$\sum_{k=1}^{\infty} \Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]\alpha_{k+p} < \beta(\alpha+1+(-1)^p) + p(-1)^p \qquad (12)$$

Proof: Let $f(z) \in S^*_{c,m,q}\, T_p(z)$, then

$$\left|\frac{z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} - p\right| < \beta\left|\frac{\alpha z\left(H_{m,q}(a_1)f(z)\right)'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} + p\right|$$

Using (4), that is

$$H_{m,q}(a_1)f(z) = z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p} z^{k+p}$$

and

$$H_{m,q}(a_1)f(-z) = (-1)^p z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+p}$$

then we have

$$\left|z\left(H_{m,q}(a_1)f(z)\right)' - p\,H_{m,q}(a_1)f(z) + p\,H_{m,q}(a_1)f(-z)\right| < \beta\left|\alpha z\left(H_{m,q}(a_1)f(z)\right)' + p\,H_{m,q}(a_1)f(z) - p\,H_{m,q}(a_1)f(-z)\right|$$

that is

$$\left|p z^p - \sum_{k=1}^{\infty}(p+k)\Gamma_k(a_1)\alpha_{k+p} z^{k+p} - p z^p + \sum_{k=1}^{\infty} p\,\Gamma_k(a_1)\alpha_{k+p} z^{k+p} + p(-1)^p z^p - p\sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+p}\right|$$
$$< \beta\left|\alpha p z^p - \sum_{k=1}^{\infty}\alpha(k+p)\Gamma_k(a_1)\alpha_{k+p} z^{k+p} + p z^p - \sum_{k=1}^{\infty} p\,\Gamma_k(a_1)\alpha_{k+p} z^{k+p} - p(-1)^p z^p + p\sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}(-1)^{k+p} z^{k+p}\right|$$

Which readily gives

$$\left|p(-1)^p z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p} z^{k+p}[k + p(-1)^{k+p}]\right|$$
$$< \beta\left|(\alpha+1-(-1)^p)p z^p - \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p} z^{k+p}[\alpha(k+p) + p - p(-1)^{k+p}]\right|$$

also,

$$|p(-1)^p z^p| + \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}|z|^{k+p}[k + p(-1)^{k+p}]$$
$$< \beta|(\alpha+1-(-1)^p)p z^p| - \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}|z|^{k+p}[\beta\alpha(k+p) + \beta p - \beta p(-1)^{k+p}]$$

that is

$$-p(-1)^p|z^p| + \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}|z|^{k+p}[k + p(-1)^{k+p}] + \sum_{k=1}^{\infty}\Gamma_k(a_1)\alpha_{k+p}|z|^{k+p}[\beta\alpha(k+p) + \beta p - \beta p(-1)^{k+p}]$$
$$< \beta(\alpha+1-(-1)^p)p|z|^p$$

Which gives

$$\sum_{k=1}^{\infty}\Gamma_k(a_1)[k(1+\beta\alpha) + p((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p}))]\alpha_{k+p}|z|^{k+p} < [\beta(\alpha+1-(-1)^p)p + p(-1)^p]|z|^p$$

Let |z| → 1⁻. So we have

$$\sum_{k=1}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) + p((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p}))]\alpha_{k+p} < \beta(\alpha+1+(-1)^p) + p(-1)^p$$

Hence by maximum modulus theore, we have $f(z) \in S^*_{c,m,q}\, T_p(z)$.

Conversely: - we assume that

$$\left| \frac{\frac{z(H_{m,q}(a_1)f(z))'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} - p}{\frac{\alpha\, z(H_{m,q}(a_1)f(z))'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} + p} \right| < \beta$$

That is

$$\left| \frac{p(-1)^p z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p} z^{k+p}[k+p(-1)^{k+p}]}{(\alpha+1-(-1)^p)p\, z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}[\alpha(k+p)+p-p(-1)^{k+p}]} \right| < \beta$$

Now since $Re\, f(z) \le |f(z)|$ for all z we have

$$Re\left\{ \frac{p(-1)^p z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p} z^{k+p}[k+p(-1)^{k+p}]}{(\alpha+1-(-1)^p)p\, z^p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}\, z^{k+p}[\alpha(k+p)+p-p(-1)^{k+p}]} \right\} < \beta \tag{13}$$

On the real axis choose values of z, we have $\frac{z(H_{m,q}(a_1)f(z))'}{H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z)} - p$ is real and $H_{m,q}(a_1)f(z) - H_{m,q}(a_1)f(-z) \neq 0$ for z ≠ 0. Upon clearing denominator in (13) and letting z →1⁻ along the real values, we

$$-p(-1)^p + \sum_{k=1}^{\infty} \Gamma_k(a_1)\alpha_{k+p}[k+p(-1)^{k+p}] < \beta(\alpha+1-(-1)^p)p - \sum_{k=1}^{\infty} \Gamma_k(a_1)\,\alpha_{k+p}[\beta\alpha(k+p) + \beta p - \beta p(-1)^{k+p}]$$

This completes the proof of Theorem 1. ▪

Corollary 1: Let the function f (z) defined by (8) be in the class $S^*_{s,m,q}\, T_p(z)$. Then we have

$$\alpha_{k+p} \le \frac{\beta(\alpha+1+(-1)^p) + p(-1)^p}{\Gamma_k(a_1)[k(1+\beta\alpha) + p((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p}))]} \quad (k \ge 1). \tag{14}$$

The equality in (14) is attained for the function f (z) given by

$$f(z) = z^p - \frac{\beta(\alpha+1+(-1)^p) + p(-1)^p}{\Gamma_k(a_1)[k(1+\beta\alpha) + p((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p}))]} z^k \quad (k \ge 1). \tag{15}$$

Theorem 2: Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\bar{z})} \neq 0$ for z ≠ 0. Then f (z) is in the class $S^*_{c,m,q}\, T_p(z)$ if and only if

$$\sum_{k=2}^{\infty} \Gamma_k(a_1)[k(1+\beta\alpha) - p[1-\beta(\alpha+2)]]\alpha_{k+p} < p\beta(\alpha+2) - p. \tag{16}$$

Corollary 2: Let the function f (z) defined by (8) be in the class $S^*_{c,m,q}\, T_p(z)$. Then we have

$$a_{k+p} \leq \frac{p\beta(\alpha+2)-p}{\Gamma_k(a_1)[k(1+\beta\alpha)-p\,[1-\beta(\alpha+2)]]} \quad (k \geq 1). \tag{17}$$

The equality in (17) is attained for the function f (z) given by

$$f(z) = z^p - \frac{p\beta(\alpha+2)-p}{\Gamma_k(a_1)[k(1+\beta\alpha)-p\,[1-\beta(\alpha+2)]]}\, z^{k+p} \quad (k \geq 1). \tag{18}$$

Theorem 3. Let the function f be defined by (8) and $H_{m,q}(a_1)f(z) - \overline{H_{m,q}(a_1)f(\overline{z})} \neq 0$ for z ≠ 0. Then f (z) is in the class $S^*_{sc,m,q}\,T_p(z)$ if and only if

$$\sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p} + \beta(\alpha+1-(-1)^{k+p})\right)\right]a_{k+p} < \beta(\alpha+1+(-1)^p) + p(-1)^p \tag{19}$$

Corollary 3: Let the function f (z) defined by (8) be in the class $S^*_{sc,m,q}\,T_p(z)$. Then we have

$$a_{k+p} \leq \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]} \quad (k \geq 1). \tag{20}$$

The equality in (20) is attained for the function f (z) given by

$$f(z) = z^p - \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{\Gamma_k(a_1)\left[k(1+\beta\alpha)+p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]}\, z^k \quad (k \geq 1) \tag{21}$$

3. Growth and Distortion Bounds

Theorem 4. Let the function f defined by (8) be in the class $S^*_{s,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq r^p - \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\, r^{p+1} \tag{22}$$

and

$$|f(z)| \leq r^p + \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\, r^{p+1} \tag{23}$$

for $z \in U$. The equalities in (22) and (23) are attained for the function given by

$$|f(z)| = z^p - \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\cdot z^{p+1} \tag{24}$$

at z = r.

Proof:- Since for k ≥ 1

$$1+\beta\alpha + p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right) \leq \Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]$$

using Theorem 1, we have

$$\left\{1+\beta\alpha + p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)\right\}\sum_{k=1}^{\infty} a_{k+p}$$
$$\leq \sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha) + p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right]a_{k+p} \leq \beta(\alpha+1+(-1)^p)+p(-1)^p$$

Then

$$\sum_{k=1}^{\infty} a_{k+p} \leq \frac{\beta(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)} \tag{25}$$

From (25) and (8), we have

$$|f(z)| \geq r^p - r^{p+1}\sum_{k=1}^{\infty} a_{k+p} \geq r^p - \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^{p+1}$$

And

$$|f(z)| \leq r^p + r^{p+1}\sum_{k=1}^{\infty} a_{k+p} \leq r^p + \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^{p+1}$$

This completes the proof of Theorem 4.

Theorem 5:- Let the function f(z) defined by (8) be in the $S^*_{s,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq pr^{p-1} - \frac{\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}(1+p)}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^p \tag{26}$$

And

$$|f'(z)| \leq pr^{p-1} + \frac{\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}(1+p)}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^p \tag{27}$$

for $z \in U$. The equalities in (22) and (23) are attained for the function f(z) given by (24).

Proof: - for k ≥ 1. Using theorem 1, we have

$$\begin{aligned}\left\{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)\right\}\sum_{k=1}^{\infty}(k+p)a_{k+p} \\ \leq (1+p)\sum_{k=1}^{\infty}\Gamma_k(a_1)\left[k(1+\beta\alpha)+\ p\left((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\right)\right] a_{k+p} \\ \leq (1+p)\,\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}\end{aligned}$$

Then

$$\sum_{k=1}^{\infty}(k+p)a_{k+p} \leq \frac{(1+p)\,\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)} \tag{28}$$

From (28) & (8), we have

$$|f'(z)| \geq pr^{p-1} - r^p\sum_{k=1}^{\infty}(k+p)a_{k+p} \geq pr^{p-1} - \frac{\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}(1+p)}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^p$$

And

$$|f'(z)| \leq pr^{p-1} + r^p\sum_{k=1}^{\infty}(k+p)a_{k+p} \leq pr^{p-1} + \frac{\{\beta\,(\alpha+1+(-1)^p)+p(-1)^p\}(1+p)}{1+\beta\alpha+\ p\left((-1)^{1+p}+\beta(\alpha+1-(-1)^{1+p})\right)}\,r^p$$

This completes the proof of Theorem 5.

On similar lines of Theorem 4 and Theorem 5, we can easily prove the following Theorem 6 and Theorem 7 respectively for f (z) belongs to $S^*_{c,m,q}\,T_p(z)$.

Theorem 6. Let the function f defined by (8) be in the class $S^*_{c,m,q}\,T_p(z)$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq r^p - \frac{p\beta(\alpha+2)-p}{(1+\beta\alpha)\ -p\,[1-\beta(\alpha+2)]}\,.\,r^{p+1} \tag{29}$$

and

$$|f(z)| \leq r^p + \frac{p\beta(\alpha+2)-p}{(1+\beta\alpha)\ -p\,[1-\beta(\alpha+2)]}\,.\,r^{p+1} \tag{30}$$

for $z \in U$. The equalities in (29) and (30) are attained for the function given by

$$|f(z)| = z^p - \frac{p\beta(\alpha+2)-p}{(1+\beta\alpha)\ -p\,[1-\beta(\alpha+2)]}\,.\,z^{p+1} \tag{31}$$

at z = r.

Theorem 7 :- Let the function f(z) defined by (8) be in the $S^{*}_{c,m,q}\, T_p(z)$. Then for $|z| = r < 1$, we have

$$|f'(z)| \geq pr^{p-1} - \frac{\{p\beta(\alpha+2)-p\}(1+p)}{(1+\beta\alpha)-p\,[1-\beta(\alpha+2)]}\, r^p \tag{32}$$

and

$$|f'(z)| \leq pr^{p-1} + \frac{\{p\beta(\alpha+2)-p\}(1+p)}{(1+\beta\alpha)-p\,[1-\beta(\alpha+2)]}\, r^p \tag{33}$$

for $z \in U$. The equalities in (32) and (33) are attained for the function f(z) given by (31).

4. Extreme Points

Theorem 8:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{k(1+\beta\alpha)+\,p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)}\, z^{k+p} \tag{34}$$

where k ≥ 1. Then $f(z) \in S^{*}_{s,m,q}\, T_p(z)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z) \tag{35}$$

Where $\Upsilon_{k+p} \geq 0\ (k \geq 1)$ and $\sum_{k=0}^{\infty} \Upsilon_{k+p} = 1$.

Proof: Suppose

$$f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p} f_{k+p}(z) = z^p - \sum_{k=1}^{\infty} \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{k(1+\beta\alpha)+\ p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)}\, \Upsilon_{k+p}\, z^{k+p}$$

Then we get

$$\sum_{k=1}^{\infty} \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{k(1+\beta\alpha)+\ p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)} \cdot \frac{k(1+\beta\alpha)+\ p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)}{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}\, \Upsilon_k = \sum_{k=1}^{\infty} \Upsilon_{k+p} = 1-\Upsilon_p \leq 1.$$

it follows from Theorem 1 that the function $f(z) \in S^{*}_{s,m,q}\, T_p(z)$.

Conversely: suppose that $f(z) \in S^{*}_{s,m,q}\, T_p(z)$. Again by using Theorem 1, we can show that

$$|\alpha_{k+p}| \leq \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{k(1+\beta\alpha)+\,p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)} \quad (k \geq 1). \tag{36}$$

setting

$$|\Upsilon_{k+p}| \leq \frac{k(1+\beta\alpha)+\,p\big((-1)^{k+p}+\beta(\alpha+1-(-1)^{k+p})\big)}{\beta\,(\alpha+1+(-1)^p)+p(-1)^p} \tag{37}$$

and

$$\Upsilon_p = 1 - \sum_{k=1}^{\infty} \Upsilon_{k+p} \tag{38}$$

We can see that f(z) can be expressed in the form of (34). This completes the proof of Theorem 8.

Corollary 4: The extreme points of the class $S^{*}_{s,m,q}\, T_p(z)$ are functions $f_{k+p}(z)\ (k \geq 1)$ given by Theorem 8.

Similar to Theorem 8, we can easily prove the following theorems for $f(z) \in S^{*}_{c,m,q}\, T_p(z)$ and $f(z) \in S^{*}_{sc,m,q}\, T_p(z)$ classes.

Theorem 9:- Theorem 8:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{p\beta(\alpha+2)-p}{k(1+\alpha\beta)-p[1-\beta(\alpha+2)]}\, z^{k+p} \tag{39}$$

where k ≥ 1. Then $f(z) \in S^*_{s,m,q}\, T_p(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z)$ $\Upsilon_{k+p} \geq 0\ (k \geq 1)$ and $\sum_{k=0}^{\infty} \Upsilon_{k+p} = 1$.

Corollary 5: The extreme points of the class $S^*_{c,m,q}\, T_p(z)$ are functions $f_{k+p}(z)\ (k \geq 1)$ given by Theorem 9.

Theorem 10:- Let $f_p(z) = z^p$ and

$$f_{k+p}(z) = z^p - \frac{\beta\,(\alpha+1+(-1)^p)+p(-1)^p}{k(1+\beta\alpha)+\ p\big((-1)^{k+p}+\beta\big(\alpha+1-(-1)^{k+p}\big)\big)}\ z^{k+p} \tag{40}$$

where k ≥ 1. Then $f(z) \in S^*_{sc,m,q}\, T_p(z)$ if and only if it can be expressed in the form $f(z) = \sum_{k=0}^{\infty} \Upsilon_{k+p}\, f_{k+p}(z)$ $\Upsilon_{k+p} \geq 0\ (k \geq 1)$ and $\sum_{k=0}^{\infty} \Upsilon_{k+p} = 1$.

Corollary 4: The extreme points of the class $S^*_{sc,m,q}\, T_p(z)$ are functions $f_{k+p}(z)\ (k \geq 1)$ given by Theorem 10.

5. Particular Cases

We have the following interesting relationships with some of the special function classes for suitable choices of parameters, which were investigated recently:

1. For p=1, q = 2, s =1, $a_1 = \lambda + 1$ ($\lambda > -1$) and $a_2 = b_1 = 1$ the class $H_{m,q}(a_1)f(z)$ reduces to the class $D^\lambda f(z)$, where D^λ ($\lambda > -1$) is theRuscheweyh derivative operator (see [10] and [13]).
2. For p=1, q = 2, s = 1, $a_1 = 2$, $a_2 = 1$ and $b_1 = 2-\mu$ ($\mu \neq 2,3,..$) the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces $\Omega^\mu_z f(z)$. Where Ω^μ_z is the Srivastava-Owa fractional derivative operator (see [5] and [7]).
3. For p=1, q = 2, s = 1, $a_1 = 2$, $a_2 = 1$ and $b_1 = k+1(k > -1)$ the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces to $I_k f(z)$. Where I_k is the Noor integral operator (see [14]).
4. For p=1, q = 2, s = 1, $a_1 = \nu+1(\nu > -1)$, $a_2 = 1$ and $b_1 = \nu + 2$ the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces to $J_\nu f(z)$. Where J_ν is the Generalized Bernardi-Libra-Livingston operator (see [4], [6], [8]).
5. For p=1, q = 2, s = 1, $a_1 = a$ ($a > 0$), $a_2 = 1$ and $b_1 = c$ ($c > 0$) the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces to L(a,c). Where L(a,c) is the Carlson-Shaffer Operator (see [12]).
6. For p=1, q = 2, s = 1, $a_1 = \mu$ ($\mu > 0$), $a_2 = 1$ and $b_1 = \lambda+1$ ($\lambda > -1$) the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces $I_{\lambda,\mu} f(z)$. Where $I_{\lambda,\mu}$ is the Choi-Saigo-Srivastava Operator (see [3]).
7. For p=1, q = 2 and s = 1 the Dziok-Srivastava Operator $H_{m,q}(a_1)f(z)$ reduces to the Hohlow Operator $I^{a_1,a_2}_{b_1}$, (see [9]).

References

1. Robertson MS. on the theory of univalent functions Ann. Math. 1936; 37:374-408.
2. Dziok J, Srivastava HM. Classes of analytic functions associated with the generalized hyper geometric function, Appl. Math. Comput. 1999; 103:1-13.
3. Choi JH, Siago M, Srivastava HM. Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 2002; 276:432-445.
4. Bernardi SD. Convex and starlike univalent functions Trans. Amer. Math. Soc. 1969; 135:429-449.
5. Owa S. On the distortion theorems. I, Kyungpook Math. J. 1969; 18(1978):53-59.
6. Libere RI. Some classes of regular univalent functions, Proc. Amer. Math. Soc. 1969; 16:755-758.
7. Owa S, Srivastava HM. Univalent and starlike generalized hypergeometric functions, Canad. J. Math. 1987; 39:1057-1077.
8. Livingston AE. On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 1966; 17:352-357.
9. Hohlow Yu E. Operators and operations in the class of univalent functions Izvestiya Vysshikh Uchebnykh Zavedenii Matematika (Russian). 1978; 10:83-89.
10. Khairnar SM, Rajas SM. On a new class of analytic and univalent functions with negative coefficients defined by Ruscheweyh derivative, Int. J. Contemp. Math. Sci. 2009; 4(34):1653-1664.
11. Nasr MA, Aouf MK. Radius of convexity for the class of starlike functions of convex order, Bulletin of the faculty of science. A Physics and Mathematics. 1983, 153-159.
12. Carlson RC, Shaffer DR. Starlike and prestralike hypergeometric functions, J. Math. Anal. Appl. 2004; 292:470-483
13. Ruscheweyh St. New Criteria for univalent functions, Proc. Amer. Math. Soc. 1975; 49:109-115.
14. Noor KI. On new classes of integral operators, J. Natur. Geom. 1999; 16:71-80.

15. Altintaşandö O, özkan. Starlike, Convex and close to convex functions of complex order, Hacettepe Bulletin of Natural sciences and Engineering. Series B. 1991; 28:37-4.
16. Aouf MK, Mostafa AK, Lashin AY, Munassar BM. On a new class of univalent functions with negative coefficients defined by Dziok-Srivastava Operator, J. Math. Anal. Appl. 2014; 2:117-131.