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On the homogeneous ternary quadratic equation $5x^2 – 2y^2 = 18z^2$

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Abstract
The quadratic diophantine equation with three unknowns represented by $5x^2 – 2y^2 = 18z^2$ is analyzed for finding its non-zero distinct integral solutions. Different patterns of solutions of the equation under consideration are obtained.

Keywords: ternary quadratic equation with three unknowns, integral solutions

1. Introduction
The quadratic diophantine equation with three unknowns offers an unlimited field for research because of their variety [1-3]. In particular, one may refer [4-19] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $5x^2 - 2y^2 = 18z^2$ representing homogeneous quadratic diophantine equation with three unknowns for determining its infinitely many non-zero integral solutions.

2. Method of Analysis
The ternary quadratic equation under consideration is

$$5x^2 - 2y^2 = 18z^2 \qquad (1)$$

Introducing the linear transformations

$$x = X + 2T, y = X + 5T \qquad (2)$$

in (1), it leads to

$$X^2 = 10T^2 + 6z^2 \qquad (3)$$

Again introducing the linear transformations

$$T = \alpha + 6\beta \ , \ z = \alpha - 10\beta \ , \ X = 4w \qquad (4)$$

in (3), it is written as

$$w^2 = \alpha^2 + 60\beta^2 \qquad (5)$$

Pattern: 1
Equation (5) is satisfied by

$$\alpha = 60r^2 - s^2 \ , \ \beta = 2rs \ , \ w = 60r^2 + s^2 \qquad (6)$$

Substituting (6) in (4) we get,

$$T = 60r^2 - s^2 + 12rs \ , \ X = 240r^2 + 4s^2$$

In view of (2), we have

$$\left.\begin{aligned} x &= 2s^2 + 360r^2 + 24rs \\ y &= 540r^2 - s^2 + 60rs \\ z &= 60r^2 - s^2 - 20rs \end{aligned}\right\} \qquad (7)$$

Thus (7) represents non-zero distinct integer solutions of (1)

Remark
Note that (2) can be represented as

$$x = X - 2T, y = X - 5T$$

Also in (4), the values of T and Z are taken as

$$T = \alpha - 6\beta, z = \alpha + 10\beta$$

In this case, one obtains three more choices of integer solutions of (1), and for simplicity, these choices are exhibited below:

Choice 1

$$x = 4w + 2(\alpha - 6\beta) = 360r^2 + 2s^2 - 24rs$$
$$y = 4w + 5(\alpha - 6\beta) = 240r^2 - s^2 - 60rs$$
$$z = \alpha + 10\beta = 60r^2 - s^2 + 20rs$$

Choice 2

$$x = 4w - 2(\alpha + 6\beta) = 120r^2 + 6s^2 - 24rs$$
$$y = 4w - 5(\alpha + 6\beta) = -60r^2 + 9s^2 - 60rs$$
$$z = \alpha - 10\beta = 60r^2 - 2s^2 - 20rs$$

Choice 3

$$x = 4w - 2(\alpha - 6\beta) = 120r^2 + 6s^2 + 24rs$$
$$y = 4w - 5(\alpha - 6\beta) = -60r^2 + 9s^2 + 60rs$$
$$z = \alpha + 10\beta = 60r^2 - 2s^2 + 20rs$$

Pattern 2
One may write (5) in the form of ratio as

$$\frac{(w+\alpha)}{60\beta} = \frac{\beta}{(w-\alpha)} = \frac{r}{s}\ ,\ s \neq 0 \qquad (8)$$

Which is equivalent to the system of double equations

$$ws + s\alpha - 60\beta r = 0$$
$$-sw - s\alpha + \beta s = 0$$

Applying the method of cross multiplication we have

$$w = s^2 + 60rs\ ,\ \alpha = 60rs - s^2\ ,\ \beta = 2s^2$$

The corresponding non-zero distinct integer solutions to (1) are given by

$$x = 26s^2 + 360rs$$
$$y = 59s^2 + 540rs$$
$$z = 60rs - 21s^2$$

Note: 1
Instead of (5), write (8) as

$$\frac{(w+\alpha)}{4\beta} = \frac{15\beta}{(w-\alpha)} = \frac{r}{s}\ ,\ s \neq 0 \qquad (9)$$

Which is equivalent to the system of double equations

$$ws + s\alpha - 60\beta r = 0$$
$$-sw - s\alpha + \beta s = 0$$

Applying the method of cross multiplication we have

$$w = 15s^2 + 4r^2 \ , \ \alpha = 4r^2 - 15s^2 \ , \ \beta = 2sr$$

The corresponding non-zero distinct integer solutions to (1) are given by

$$x = 30s^2 + 24r^2 + 24rs$$
$$y = -15s^2 + 36r^2 + 60rs$$
$$z = -15s^2 + 4r^2 - 20rs$$

Note: 2
Instead of (9), write (5) as

$$\frac{(w+\alpha)}{2\beta} = \frac{30\beta}{(w-\alpha)} = \frac{r}{s}, s \neq 0 \qquad (10)$$

which is equivalent to the system of double equations

$$ws + s\alpha - 2\beta r = 0$$
$$-sw - s\alpha + 30\beta s = 0$$

Applying the method of cross multiplication we have

$$w = 30s^2 + 2r^2 \ , \ \alpha = 2r^2 - 30s^2 \ , \ \beta = 2rs$$

The corresponding non-zero distinct integer solution to (1) are given by

$$x = 60s^2 + 12r^2 + 24rs$$
$$y = -30s^2 + 18r^2 + 60rs$$
$$z = -30s^2 + 2r^2 - 20rs$$

Pattern 3
Note that (9) is expressed in the system of double equations as follows:

Table 1

$w+\alpha$	$60\beta^2$	$30\beta^2$	$15\beta^2$	60β	30β	15β
$w-\alpha$	1	2	4	β	2β	4β

Solving each of the above systems, the corresponding solutions to (1) are given below:

Solution for System 1
Case (i) $\beta = 2r$

$$x = 360r^2 + 24r + 2 \ , \ y = 540r^2 + 60r - 1 \ , \ z = 60r^2 - 1 - 20rs$$

Case (ii) $\beta = 2r + 1$

$$x = 360r^2 + 384r + 104 \ , \ y = 540r^2 + 600r + 164 \ , \ z = 60r^2 + 40r + 4$$

Solution for system 2

$$x = 180r^2 + 24r + 4 \ , \ y = 270r^2 + 60r - 2 \ , \ z = 30r^2 - 2 - 20rs$$

Solution for system 3

$$x = 386r \ , \ y = 599r \ , \ z = 39r$$

Solution for system 4

Case (i) $\beta = 2r$

$$x = 208r\ ,\ y = 328r\ ,\ z = 8r$$

Case (ii) $\beta = 2r + 1$

$$x = 208r + 104\ ,\ y = 328r + 164\ ,\ z = 8r + 4$$

Solution for System 5

$$x = 122r\ ,\ y = 191r\ ,\ z = -9r$$

Pattern 4

One may write (5) as

$$\alpha^2 + 60\beta^2 = w^2 * 1 \tag{11}$$

Assume $w = a^2 + 15b^2$ (12)

Write 1 as

$$1 = \frac{\left(1 + i\sqrt{15}\right)\left(1 - i\sqrt{15}\right)}{16} \tag{13}$$

Using (12) and (13) in (11) and employing the method of factorization, define

$$\alpha + i2\sqrt{15}\beta = \left(a + i\sqrt{15}b\right)^2 \frac{\left(1 + i\sqrt{15}\right)}{4}$$

Equating real and imaginary parts, we get

$$\alpha = 4a^2 - 60b^2 - 120ab\ ,\ \beta = 2a^2 - 30b^2 + 4ab \tag{14}$$

Substituting (14) in (4), we get

$$T = 4a^2 - 60b^2 - 120ab\ ,\ X = 64a^2 + 960b^2$$

In view of (2), the corresponding non-zero distinct integer solutions to (1) are given by

$$x = 96a^2 + 480b^2 + 192ab$$

$$y = 144a^2 - 240b^2 - 480ab$$

$$z = -16a^2 + 240b^2 - 160ab$$

3. Conclusion

In this paper, we have made an attempt to obtain infinitely many non-zero distinct integer solutions to the equation given by $5x^2 - 2y^2 = 18z^2$. As ternary quadratic equations are rich in variety, one may search for the other choice of ternary quadratic Diophantine equations and determine their integer solutions along with suitable properties.

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