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On the positive Pell equation $y^2 = 14x^2 + 18$

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Abstract
Non-homogeneous binary quadratic equation representing hyperbola given by $y^2 = 14x^2 + 18$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbola and parabola are presented. Also, employing the solutions of the given equation, special Pythagorean triangle is constructed.

Keywords: binary quadratic, hyperbola, parabola, positive Pell equation, integral solution

1. Introduction

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as Pell equation and is one of the oldest Diophantine equation that has interested mathematicians all over all the world, since antiquity, J.L. Lagrange proved that all positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions. In [3] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 14$. In [4] a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [5] different patterns of infinitely many Pythagorean triangles are obtained by employing the non-integral solutions of $y^2 = 12x^2 + 1$. In this context one may refer [6-20]. In this communication, the positive Pell equation given by $y^2 = 14x^2 + 18$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solution are presented. Also knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented. Also employing the solutions of the given equation, special Pythagorean triangle is constructed.

2. Method of analysis

The positive Pell equation representing, hyperbola under consideration is

$$y^2 = 14x^2 + 18 \tag{1}$$

whose smallest positive integer solution (x_0, y_0) of (1) is $x_0 = 3, y_0 = 12$ To obtain the other solutions of (1), consider the Pellian equation

$$y^2 = 14x^2 + 1 \tag{2}$$

whose smallest positive integer solutions is

$\tilde{x}_0 = 4$, $\tilde{y}_0 = 15$ The general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by\

$$\tilde{x}_n = \frac{1}{2\sqrt{14}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n \tag{3}$$

where

$$f_n = \left(15 + 4\sqrt{14}\right)^{n+1} + \left(15 - 4\sqrt{14}\right)^{n+1}, n = -1, 0, 1 \ldots$$

$$g_n = \left(15+4\sqrt{14}\right)^{n+1} - \left(15-4\sqrt{14}\right)^{n+1}, n = -1,0,1\ldots$$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = x_0\tilde{y}_n + y_0\tilde{x}_n = \frac{3}{2} f_n + \frac{12}{2\sqrt{14}} g_n \quad (4)$$

$$y_{n+1} = y_0\tilde{y}_n + 14x_0\tilde{x}_n = 12 f_n + 3\sqrt{14} g_n \quad (5)$$

Thus (4) and (5) represent the integer solutions of the hyperbola (1).
Recurrence relations for x and y are:

$$x_{n+3} - 30x_{n+2} + x_{n+1} = 0, \; n = -1,0,1\ldots$$

$$y_{n+3} - 30y_{n+2} + y_{n+1} = 0, \; n = -1,0,1\ldots$$

A few numerical values are given in the following Table: 1 below

Table 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	3	12
0	93	348
1	2787	10428
2	83517	312492

2.1 A few interesting relations among the solutions are given below

1. $15x_{n+1} - x_{n+2} + 4y_{n+1} = 0$
2. $x_{n+1} - 15x_{n+2} + 4y_{n+2} = 0$
3. $15x_{n+1} - 449x_{n+2} + 4y_{n+3} = 0$
4. $449x_{n+1} - x_{n+3} + 120y_{n+1} = 0$
5. $x_{n+1} - x_{n+3} + 8y_{n+2} = 0$
6. $x_{n+1} - 449x_{n+3} + 120y_{n+3} = 0$
7. $56x_{n+1} + 15y_{n+1} - y_{n+2} = 0$
8. $1680x_{n+1} + 449y_{n+1} - y_{n+3} = 0$
9. $56x_{n+1} + 449y_{n+2} - 15y_{n+3} = 0$
10. $449x_{n+2} - 15x_{n+3} + 4y_{n+1} = 0$
11. $15x_{n+2} - x_{n+3} + 4y_{n+2} = 0$
12. $x_{n+2} - 15x_{n+3} + 4y_{n+3} = 0$
13. $56x_{n+2} + y_{n+1} - 15y_{n+2} = 0$
14. $112x_{n+2} + y_{n+1} - y_{n+3} = 0$
15. $56x_{n+2} + 15y_{n+2} - y_{n+3} = 0$
16. $56x_{n+3} + 15y_{n+1} - 449y_{n+2} = 0$
17. $1680x_{n+3} + y_{n+1} - 449y_{n+3} = 0$

18. $56x_{n+3} + y_{n+2} - 15y_{n+3} = 0$

2.2 Each of following expressions represents a cubic integer

1. $\frac{1}{72}\left[\left(-696x_{3n+3} + 24x_{3n+4}\right) + \left(-2088x_{n+1} + 72x_{n+2}\right)\right]$
2. $\frac{1}{2160}\left[\left(-20856x_{3n+3} + 24x_{3n+5}\right) + \left(-62568x_{n+1} + 72x_{n+3}\right)\right]$
3. $\frac{1}{18}\left[\left(-84x_{3n+3} + 24y_{3n+3}\right) + \left(-252x_{n+1} + 72y_{n+1}\right)\right]$
4. $\frac{1}{270}\left[\left(-2604x_{3n+3} + 24y_{3n+4}\right) + \left(-7812x_{n+1} + 72y_{n+2}\right)\right]$
5. $\frac{1}{8082}\left[\left(-78036x_{3n+3} + 24y_{3n+5}\right) + \left(-234108x_{n+1} + 72y_{n+3}\right)\right]$
6. $\frac{1}{72}\left[\left(-20856x_{3n+4} + 696x_{3n+5}\right) + \left(-62568x_{n+2} + 2088x_{n+3}\right)\right]$
7. $\frac{1}{270}\left[\left(-84x_{3n+4} + 696y_{3n+3}\right) + \left(-252x_{n+2} + 2088y_{n+1}\right)\right]$
8. $\frac{1}{18}\left[\left(-2604x_{3n+4} + 696y_{3n+4}\right) + \left(-7812x_{n+2} + 2088y_{n+2}\right)\right]$
9. $\frac{1}{270}\left[\left(-78036x_{3n+4} + 696y_{3n+5}\right) + \left(-234108x_{n+2} + 2088y_{n+3}\right)\right]$
10. $\frac{1}{8082}\left[\left(-84x_{3n+5} + 20856y_{3n+3}\right) + \left(-252x_{n+3} + 62568y_{n+1}\right)\right]$
11. $\frac{1}{270}\left[\left(-2604x_{3n+5} + 20856y_{3n+4}\right) + \left(-7812x_{n+3} + 62568y_{n+2}\right)\right]$
12. $\frac{1}{18}\left[\left(-78036x_{3n+5} + 20856y_{3n+5}\right) + \left(-234108x_{n+3} + 62568y_{n+3}\right)\right]$
13. $\frac{1}{72}\left[\left(186y_{3n+3} - 6y_{3n+4}\right) + \left(558y_{n+1} - 18y_{n+2}\right)\right]$
14. $\frac{1}{2160}\left[\left(5574y_{3n+3} - 6y_{3n+5}\right) + \left(16722y_{n+1} - 18y_{n+3}\right)\right]$
15. $\frac{1}{72}\left[\left(5574y_{3n+4} - 186y_{3n+5}\right) + \left(16722y_{n+2} - 558y_{n+3}\right)\right]$

2.3 Each of following expressions represents a bi-quadratic integer

1. $\frac{1}{(72)^2}\left[\left(-50112x_{4n+4} + 1728x_{4n+5}\right) + 4\left(-696x_{n+1} + 24x_{n+2}\right)^2 - 10368\right]$

$\frac{1}{(2160)^2}\left[\left(-45048960x_{4n+4} + 51840x_{4n+6}\right) + 4\left(-20856x_{n+1} + 24x_{n+3}\right)^2 - 9331200\right]$

2. $\frac{1}{(18)^2}\left[\left(-1512x_{4n+4} + 432y_{4n+4}\right) + 4\left(-84x_{n+1} + 24y_{n+1}\right)^2 - 648\right]$
3. $\frac{1}{(270)^2}\left[\left(-703080x_{4n+4} + 6480y_{4n+5}\right) + 4\left(-2604x_{n+1} + 24y_{n+2}\right)^2 - 145800\right]$

4. $\frac{1}{(8082)^2}\left[\left(-630686952x_{4n+4}+193968y_{4n+6}\right)+4\left(-78036x_{n+1}+24y_{n+3}\right)^2-130637448\right]$
5. $\frac{1}{(72)^2}\left[\left(-1501632x_{4n+5}+50112x_{4n+6}\right)+4\left(-20856x_{n+2}+696x_{n+3}\right)^2-10368\right]$
6. $\frac{1}{(270)^2}\left[\left(-22680x_{4n+5}+187920y_{4n+4}\right)+4\left(-84x_{n+2}+696y_{n+1}\right)^2-145800\right]$
7. $\frac{1}{(18)^2}\left[\left(-46872x_{4n+5}+12528y_{4n+5}\right)+4\left(-2604x_{n+2}+696y_{n+2}\right)^2-648\right]$
8. $\frac{1}{(270)^2}\left[\left(-21069720x_{4n+5}+187920y_{4n+6}\right)+4\left(-78036x_{n+2}+696y_{n+3}\right)^2-145800\right]$
9. $\frac{1}{(8082)^2}\left[\left(-678888x_{4n+6}+168558192y_{4n+4}\right)+4\left(-84x_{n+3}+20856y_{n+1}\right)^2-130637448\right]$
10. $\frac{1}{(270)^2}\left[\left(-703080x_{4n+6}+5631120y_{4n+5}\right)+4\left(-2604x_{n+3}+20856y_{n+2}\right)^2-145800\right]$
11. $\frac{1}{(18)^2}\left[\left(-1404648x_{4n+6}+375408y_{4n+6}\right)+4\left(-78036x_{n+3}+20856y_{n+3}\right)^2-648\right]$
12. $\frac{1}{(72)^2}\left[\left(13392y_{4n+4}-432y_{4n+5}\right)+4\left(186y_{n+1}-6y_{n+2}\right)^2-10368\right]$
13. $\frac{1}{(2160)^2}\left[\left(12039840y_{4n+4}-12960y_{4n+6}\right)+4\left(5574y_{n+1}-6y_{n+3}\right)^2-9331200\right]$
14. $\frac{1}{(72)^2}\left[\left(403128y_{4n+5}-13392y_{4n+6}\right)+4\left(5574y_{n+2}-186y_{n+3}\right)^2-10368\right]$

2.4 Each of the following expressions represents a nasty number

1. $\frac{1}{72}\left[-4176x_{2n+2}+144x_{2n+3}+864\right]$
2. $\frac{1}{2160}\left[-125136x_{2n+2}+144x_{2n+4}+25920\right]$
3. $\frac{1}{18}\left[-504x_{2n+2}+144y_{2n+2}+216\right]$
4. $\frac{1}{270}\left[-15624x_{2n+2}+144y_{2n+3}+3240\right]$
5. $\frac{1}{8082}\left[-468216x_{2n+2}+144y_{2n+4}+96984\right]$
6. $\frac{1}{72}\left[-125136x_{2n+3}+4176x_{2n+4}+864\right]$
7. $\frac{1}{270}\left[-504x_{2n+3}+4176y_{2n+2}+3240\right]$
8. $\frac{1}{18}\left[-15624x_{2n+3}+4176y_{2n+3}+216\right]$
9. $\frac{1}{270}\left[-468212x_{2n+3}+4176y_{2n+4}+3240\right]$

10. $\frac{1}{8082}\left[-504x_{2n+4}+125136y_{2n+2}+96984\right]$

11. $\frac{1}{270}\left[-15624x_{2n+4}+125136y_{2n+3}+3240\right]$

12. $\frac{1}{18}\left[-468216x_{2n+4}+125136y_{2n+4}+216\right]$

13. $\frac{1}{72}\left[1116y_{2n+2}-36y_{2n+3}+864\right]$

14. $\frac{1}{2160}\left[33444y_{2n+2}-36y_{2n+4}+25920\right]$

15. $\frac{1}{72}\left[33444y_{2n+3}-1116y_{2n+4}+864\right]$

2.5 Each of the following expressions represents a quintic integer

1. $\frac{1}{(72)^3}\left[\begin{array}{l}(-3608064x_{5n+5}+124416x_{5n+6})+5(-696x_{n+1}+24x_{n+2})^3\\-(-18040320x_{n+1}+622080x_{n+2})\end{array}\right]$

2. $\frac{1}{(2160)^3}\left[\begin{array}{l}(-97305753600x_{5n+5}+111974400x_{5n+7})+5(-20856x_{n+1}+24x_{n+3})^3\\-(-486528768000x_{n+1}+559872000x_{n+3})\end{array}\right]$

3. $\frac{1}{(18)^3}\left[\begin{array}{l}(-27216x_{5n+5}+7776y_{5n+5})+5(-84x_{n+1}+24y_{n+1})^3\\-(-136080x_{n+1}+38880y_{n+1})\end{array}\right]$

4. $\frac{1}{(270)^3}\left[\begin{array}{l}(-189831600x_{5n+5}+1749600y_{5n+6})+5(-2604x_{n+1}+24y_{n+2})^3\\-(-949158000x_{n+1}+8748000y_{n+2})\end{array}\right]$

5. $\frac{1}{(8082)^3}\left[\begin{array}{l}(-5097211946064x_{5n+5}+1567649376y_{5n+7})+5(-78036x_{n+1}+24y_{n+3})^3\\-(-25486059730320x_{n+1}+7838246880y_{n+3})\end{array}\right]$

6. $\frac{1}{(72)^3}\left[\begin{array}{l}(-108117504x_{5n+6}+3608064x_{5n+7})+5(-20856x_{n+2}+696x_{n+3})^3\\-(-540587520x_{n+2}+18040320x_{n+3})\end{array}\right]$

7. $\frac{1}{(270)^3}\left[\begin{array}{l}(-6123600x_{5n+6}+50738400y_{5n+5})+5(-84x_{n+2}+696y_{n+1})^3\\-(-30618000x_{n+2}+253692000y_{n+1})\end{array}\right]$

8. $\frac{1}{(18)^3}\left[\begin{array}{l}(-843696x_{5n+6}+225504y_{5n+6})+5(-2604x_{n+2}+696y_{n+2})^3\\-(-4218480x_{n+2}+1127520y_{n+2})\end{array}\right]$

9. $\frac{1}{(270)^3}\left[\begin{array}{l}(-5688824400x_{5n+6}+50738400y_{5n+7})+5(-78036x_{n+2}+696y_{n+3})^3\\-(-28444122000x_{n+2}+253692000y_{n+3})\end{array}\right]$

10. $\frac{1}{(8082)^3}\left[\begin{array}{l}(-5486772816x_{5n+7}+1362287307744y_{5n+5})+5(-84x_{n+3}+20856y_{n+1})^3\\-(-27433864080x_{n+3}+6811436538720y_{n+1})\end{array}\right]$

11. $\frac{1}{(270)^3}\left[\begin{array}{l}(-189831600x_{5n+7}+1520402400y_{5n+6})+5(-2604x_{n+3}+20856y_{n+2})^3\\-(-949158000x_{n+3}+7602012000y_{n+2})\end{array}\right]$

12. $\frac{1}{(18)^3}\left[\begin{array}{l}(-2528366x_{5n+7}+6757344y_{5n+7})+5(-78036x_{n+3}+20856y_{n+3})^3\\-(-126418320x_{n+3}+33786720y_{n+3})\end{array}\right]$

13. $\frac{1}{(72)^3}\left[\begin{array}{l}(964224y_{5n+5}-31104y_{5n+6})+5(186y_{n+1}-6y_{n+2})^3\\-(4821120y_{n+1}-155520y_{n+2})\end{array}\right]$

14. $\frac{1}{(2160)^3}\left[\begin{array}{l}(26006054400y_{5n+5}-27993600y_{5n+7})+5(5574y_{n+1}-6y_{n+3})^3\\-(130030272000y_{n+1}-139968000y_{n+3})\end{array}\right]$

15. $\frac{1}{(72)^3}\left[\begin{array}{l}(28895616y_{5n+6}-964224y_{5n+7})+5(5574y_{n+2}-186y_{n+3})^3\\-(144478080y_{n+2}-4821120y_{n+3})\end{array}\right]$

2.6 Remarkable Observations

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 2 below:

Table 2: Hyperbolas

S. No	Hyperbolas	(X_n, Y_n)
1.	$14X_n^2 - Y_n^2 = 290304$	$[(-696x_{n+1}+24x_{n+2}),(2604x_{n+1}-84x_{n+2})]$
2.	$14X_n^2 - Y_n^2 = 261273600$	$[(-20856x_{n+1}+24x_{n+3}),(78036x_{n+1}-84x_{n+3})]$
3.	$14X_n^2 - Y_n^2 = 18144$	$[(-84x_{n+1}+24y_{n+1}),(336x_{n+1}-84y_{n+1})]$
4.	$14X_n^2 - Y_n^2 = 4082400$	$[(-2604x_{n+1}+24y_{n+2}),(9744x_{n+1}-84y_{n+2})]$
5.	$14X_n^2 - Y_n^2 = 3657848544$	$[(-78036x_{n+1}+24y_{n+3}),(291984x_{n+1}-84y_{n+3})]$
6.	$14X_n^2 - Y_n^2 = 290304$	$[(-20856x_{n+2}+696x_{n+3}),(78036x_{n+2}-2604x_{n+3})]$
7.	$14X_n^2 - Y_n^2 = 4082400$	$[(-84x_{n+2}+696y_{n+1}),(336x_{n+2}-2604y_{n+1})]$
8.	$14X_n^2 - Y_n^2 = 18144$	$[(-2604x_{n+2}+696y_{n+2}),(9744x_{n+2}-2604y_{n+2})]$
9.	$14X_n^2 - Y_n^2 = 4082400$	$[(-78036x_{n+2}+696y_{n+3}),(291984x_{n+2}-2604y_{n+3})]$
10.	$14X_n^2 - Y_n^2 = 3657848544$	$[(-84x_{n+3}+20856y_{n+1}),(336x_{n+3}-78036y_{n+1})]$
11.	$14X_n^2 - Y_n^2 = 4082400$	$[(-2604x_{n+3}+20856y_{n+2}),(9744x_{n+3}-78036y_{n+2})]$
13.	$14X_n^2 - Y_n^2 = 290304$	$[(186y_{n+1}-6y_{n+2}),(-696y_{n+1}+24y_{n+2})]$
14.	$14X_n^2 - Y_n^2 = 261273600$	$[(5574y_{n+1}-6y_{n+3}),(-20856y_{n+1}+24y_{n+3})]$
15.	$14X_n^2 - Y_n^2 = 290304$	$[(5574y_{n+2}-186y_{n+3}),(-20856y_{n+2}+696y_{n+3})]$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table 3 below:

Table 3: Parabolas

S. No	Parabolas	(X_n, Y_n)
1.	$1008X_n - Y_n^2 = 290304$	$[(-696x_{2n+2}+24x_{2n+3}+144),(2604x_{n+1}-84x_{n+2})]$
2.	$30240X_n - Y_n^2 = 261273600$	$[(-20856x_{2n+2}+24x_{2n+4}+4320),(78036x_{n+1}-84x_{n+3})]$
3.	$252X_n - Y_n^2 = 18144$	$[(-84x_{2n+2}+24y_{2n+2}+36),(336x_{n+1}-84y_{n+1})]$
4.	$3780X_n - Y_n^2 = 4082400$	$[(-2604x_{2n+2}+24y_{2n+3}+540),(9744x_{n+1}-84y_{n+2})]$
5.	$113148X_n - Y_n^2 = 3657848544$	$[(-78036x_{2n+2}+24y_{2n+4}+16164),(291984x_{n+1}-84y_{n+3})]$
6.	$1008X_n - Y_n^2 = 290304$	$[(-20856x_{2n+3}+696x_{2n+4}+144),(78036x_{n+2}-2604x_{n+3})]$
7.	$3780X_n - Y_n^2 = 4082400$	$[(-84x_{2n+3}+696y_{2n+2}+540),(336x_{n+2}-2604y_{n+1})]$
8.	$252X_n - Y_n^2 = 18144$	$[(-2604x_{2n+3}+696y_{2n+3}+36),(9744x_{n+2}-2604y_{n+2})]$
9.	$3780X_n - Y_n^2 = 4082400$	$[(-78036x_{2n+3}+696y_{2n+4}+540),(291984x_{n+2}-2604y_{n+3})]$

10.	$113148X_n - Y_n^2 = 3657848544$	$[(-84x_{2n+4} + 20856y_{2n+2} + 16164),(336x_{n+3} - 78036y_{n+1})]$
11.	$3780X_n - Y_n^2 = 4082400$	$[(-2604x_{2n+4} + 20856y_{2n+3} + 540),(9744x_{n+3} - 78036y_{n+2})]$
12.	$252X_n - Y_n^2 = 18144$	$[(-78036x_{2n+4} + 20856y_{2n+4} + 36),(291984x_{n+3} - 78036y_{n+3})]$
13.	$1008X_n - Y_n^2 = 290304$	$[(186y_{2n+2} - 6y_{2n+3} + 144),(-696y_{n+1} + 24y_{n+2})]$
14.	$30240X_n - Y_n^2 = 261273600$	$[(5574y_{2n+2} - 6y_{2n+4} + 4320),(-20856y_{n+1} + 24y_{n+3})]$
15.	$1008X_n - Y_n^2 = 290304$	$[(5574y_{2n+3} - 186y_{2n+4} + 144),(-20856y_{n+2} + 696y_{n+3})]$

2.7 Generation of Pythagorean Triangle
2.7.1 Let p, q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1},\ q = x_{n+1}$$

Note that $p > q > 0$ treat p, q as the generators of Pythagorean triangle $T(X,Y,Z)$ where

$$X = 2pq,\ Y = p^2 - q^2,\ Z = p^2 + q^2,\ p > q > 0$$

Let A, P represents the area and perimeter of Pythagorean triangle. Then the following results are observed.

1. $2X - 14Y + 12Z = 36$
2. $\frac{2A}{P} = x_{n+1} * y_{n+1}$
3. $X + Y - \frac{4A}{P}$ is written as the sum of two squares.
4. $3\left(X - \frac{4A}{P}\right)$ is a Nasty number.
5. $3(Z - Y)$ is a Nasty number.

2.7.2 Let p, q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1},\ q = y_{n+1}$$

Note that $p > q > 0$ treat p, q as the generators of Pythagorean triangle $T(X,Y,Z)$ where

$$X = 2pq,\ Y = p^2 - q^2,\ Z = p^2 + q^2,\ p > q > 0$$

Let A, P represents the area and perimeter of Pythagorean triangle. In this case, the corresponding Pythagorean triangle satisfies the relation $Y + 27Z - 28X = 36$

3. Conclusion
In this paper, we have presented infinitely many integer solutions for the hyperbola; represented by positive Pell equation is given by $y^2 = 14x^2 + 18$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

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