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On negative Pell equation $y^2 = 20x^2 - 11$

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Abstract

The binary quadratic equation represented by negative Pellian $y^2 = 20x^2 - 11$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

Keywords: binary quadratic, hyperbola, parabola, Pell equation, integral solutions

1. Introduction

The binary quadratic diophantine equations (both homogeneous and non-homogeneous) are rich in variety. In [1-16] the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. This communication concerns with yet another binary quadratic equation given by $y^2 = 20x^2 - 11$. The recurrence relations satisfied by the solutions x and y are given. Also a few interesting properties among the solutions are exhibited.

2. Method of analysis

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 20x^2 - 11 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 1,\ y_0 = 3$$

To obtain the other solutions of (1), consider the pell equation

$$y^2 = 20x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{g_n}{2\sqrt{20}} \quad , \quad \tilde{y}_n = \frac{f_n}{2}$$

Where

$$f_n = \left(9 + 2\sqrt{20}\right)^{n+1} + \left(9 - 2\sqrt{20}\right)^{n+1}$$

$$g_n = \left(9 + 2\sqrt{20}\right)^{n+1} - \left(9 - 2\sqrt{20}\right)^{n+1},\quad n = -1, 0, 1, \ldots$$

Applying Brahmagupta lemma between the solutions (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{3}{2\sqrt{20}} g_n \qquad (2)$$

$$y_{n+1} = \frac{3}{2} f_n + \frac{\sqrt{20}}{2} g_n \qquad (3)$$

(3) Thus (2) and (3) represent the non-zero distinct integer solutions of (1). Recurrence relations for x and y are:

$$x_{n+3} - 18x_{n+2} + x_{n+1} = 0, \; n = -1,0,1\ldots$$

$$y_{n+3} - 18y_{n+2} + y_{n+1} = 0, \; n = -1,0,1\ldots$$

A few numerical examples are given in the following Table: 1 below:

Table: 1: Numerical Examples

n	x_{n+1}	y_{n+1}
-1	1	3
0	15	67
1	269	1203
2	4827	21587

2.1 A few interesting relations among the solutions are given below

1. $9x_{n+1} - x_{n+2} + 2y_{n+1} = 0$
2. $x_{n+1} - 9x_{n+2} + 2y_{n+2} = 0$
3. $161x_{n+1} - x_{n+3} + 36y_{n+1} = 0$
4. $x_{n+1} - x_{n+3} + 4y_{n+2} = 0$
5. $x_{n+1} - 161x_{n+3} + 36y_{n+3} = 0$
6. $40x_{n+1} + 9y_{n+1} - y_{n+2} = 0$
7. $720x_{n+1} + 161y_{n+1} - y_{n+3} = 0$
8. $440x_{n+1} + 1771y_{n+2} - 99y_{n+3} = 0$
9. $99x_{n+1} + 1771x_{n+2} + 22y_{n+3} = 0$
10. $1771x_{n+2} - 99x_{n+3} + 22y_{n+1} = 0$
11. $9x_{n+2} - x_{n+3} + 2y_{n+2} = 0$
12. $x_{n+2} - 9x_{n+3} + 2y_{n+3} = 0$
13. $40x_{n+2} + y_{n+1} - 9y_{n+2} = 0$
14. $80x_{n+2} + y_{n+1} - y_{n+3} = 0$
15. $40x_{n+2} + 9y_{n+2} - y_{n+3} = 0$
16. $440x_{n+3} + 99y_{n+1} - 1771y_{n+2} = 0$
17. $720x_{n+3} + y_{n+1} - 161y_{n+3} = 0$
18. $40x_{n+3} + y_{n+2} - 9y_{n+3} = 0$

2.2 Each of the following expression represents a cubical integer

(i) $\frac{1}{22}\left[\left(134x_{3n+3} - 6x_{3n+4}\right) + \left(402x_{n+1} - 18x_{n+2}\right)\right]$

(ii) $\frac{1}{396}\left[\left(2406x_{3n+3}-6x_{3n+5}\right)+\left(7218x_{n+1}-18x_{n+3}\right)\right]$

(iii) $\frac{1}{11}\left[\left(40x_{3n+3}-6y_{3n+3}\right)+\left(120x_{n+1}-18y_{n+1}\right)\right]$

(iv) $\frac{1}{99}\left[\left(600x_{3n+3}-6y_{3n+4}\right)+\left(1800x_{n+1}-18y_{n+2}\right)\right]$

(v) $\frac{1}{1771}\left[\left(10760x_{3n+3}-6y_{3n+5}\right)+\left(32280x_{n+1}-18y_{n+3}\right)\right]$

(vi) $\frac{1}{22}\left[\left(2406x_{3n+4}-134x_{3n+5}\right)+\left(7218x_{n+2}-402x_{n+3}\right)\right]$

(vii) $\frac{1}{99}\left[\left(40x_{3n+4}-134y_{3n+3}\right)+\left(120x_{n+2}-402y_{n+1}\right)\right]$

(viii) $\frac{1}{11}\left[\left(600x_{3n+4}-134y_{3n+4}\right)+\left(1800x_{n+2}-402y_{n+2}\right)\right]$

(ix) $\frac{1}{99}\left[\left(10760x_{3n+4}-134y_{3n+5}\right)+\left(32280x_{n+2}-402y_{n+3}\right)\right]$

(x) $\frac{1}{1771}\left[\left(40x_{3n+5}-2406y_{3n+3}\right)+\left(120x_{n+3}-7218y_{n+1}\right)\right]$

(xi) $\frac{1}{99}\left[\left(600x_{3n+5}-2406y_{3n+4}\right)+\left(1800x_{n+3}-7218y_{n+2}\right)\right]$

(xii) $\frac{1}{11}\left[\left(10760x_{3n+5}-2406y_{3n+5}\right)+\left(32280x_{n+3}-7218y_{n+3}\right)\right]$

(xiii) $\frac{1}{22}\left[\left(2y_{3n+4}-30y_{3n+3}\right)+\left(6y_{n+2}-90y_{n+1}\right)\right]$

(xiv) $\frac{1}{396}\left[\left(2y_{3n+5}-538y_{3n+3}\right)+\left(6y_{n+3}-1614y_{n+1}\right)\right]$

(xv) $\frac{1}{22}\left[\left(30y_{3n+5}-538y_{3n+4}\right)+\left(90y_{n+3}-1614y_{n+2}\right)\right]$

2.3 Each of the following expression represents a bi-quadratic integer

- $\frac{1}{(22)^2}\left[\left(2948x_{4n+4}-132x_{4n+5}\right)+4\left(134x_{n+1}-6x_{n+2}\right)^2-968\right]$
- $\frac{1}{(396)^2}\left[\left(952776x_{4n+4}-2376x_{4n+6}\right)+4\left(2406x_{n+1}-6x_{n+3}\right)^2-313632\right]$
- $\frac{1}{(11)^2}\left[\left(440x_{4n+4}-66y_{4n+4}\right)+4\left(40x_{n+1}-6y_{n+1}\right)^2-242\right]$
- $\frac{1}{(99)^2}\left[\left(59400x_{4n+4}-594y_{4n+5}\right)+4\left(600x_{n+1}-6y_{n+2}\right)^2-19602\right]$
- $\frac{1}{(1771)^2}\left[\left(19055960x_{4n+4}-10626y_{4n+6}\right)+4\left(10760x_{n+1}-6y_{n+3}\right)^2-6272882\right]$
- $\frac{1}{(22)^2}\left[\left(52932x_{4n+5}-2948x_{4n+6}\right)+4\left(2406x_{n+2}-134x_{n+3}\right)^2-968\right]$
- $\frac{1}{(99)^2}\left[\left(3960x_{4n+5}-13266y_{4n+4}\right)+4\left(40x_{n+2}-134y_{n+1}\right)^2-19602\right]$

- $\frac{1}{(11)^2}\left[(6600x_{4n+5}-1474y_{4n+5})+4(600x_{n+2}-134y_{n+2})^2-242\right]$
- $\frac{1}{(99)^2}\left[(1065240x_{4n+5}-13266y_{4n+6})+4(10760x_{n+2}-134y_{n+3})^2-19602\right]$
- $\frac{1}{(1771)^2}\left[(70840x_{4n+6}-4261026y_{4n+4})+4(40x_{n+3}-2406y_{n+1})^2-6272882\right]$
- $\frac{1}{(99)^2}\left[(59400x_{4n+6}-238194y_{4n+5})+4(600x_{n+3}-2406y_{n+2})^2-19602\right]$
- $\frac{1}{(11)^2}\left[(118360x_{4n+6}-26466y_{4n+6})+4(10760x_{n+3}-2406y_{n+3})^2-242\right]$
- $\frac{1}{(22)^2}\left[(44y_{4n+5}-660y_{4n+4})+4(2y_{n+2}-30y_{n+1})^2-968\right]$
- $\frac{1}{(396)^2}\left[(792y_{4n+6}-213048y_{4n+4})+4(2y_{n+3}-538y_{n+1})^2-313632\right]$
- $\frac{1}{(22)^2}\left[(660y_{4n+6}-11836y_{4n+5})+4(30y_{n+3}-538y_{n+2})^2-968\right]$

2.4 Each of the following expression represents a nasty number

i) $\frac{1}{22}\left[804x_{2n+2}-36x_{2n+3}+264\right]$

ii) $\frac{1}{396}\left[14436x_{2n+2}-36x_{2n+4}+4752\right]$

iii) $\frac{1}{11}\left[240x_{2n+2}-36y_{2n+2}+132\right]$

iv) $\frac{1}{99}\left[3600x_{2n+2}-36y_{2n+3}+1188\right]$

v) $\frac{1}{1771}\left[64560x_{2n+2}-36y_{2n+4}+21252\right]$

vi) $\frac{1}{22}\left[14436x_{2n+3}-804x_{2n+4}+264\right]$

vii) $\frac{1}{99}\left[240x_{2n+3}-804y_{2n+2}+1188\right]$

viii) $\frac{1}{11}\left[3600x_{2n+3}-804y_{2n+3}+132\right]$

ix) $\frac{1}{99}\left[64560x_{2n+3}-804y_{2n+4}+1188\right]$

x) $\frac{1}{1771}\left[240x_{2n+4}-14436y_{2n+2}+21252\right]$

xi) $\frac{1}{99}\left[3600x_{2n+4}-14436y_{2n+3}+1188\right]$

xii) $\frac{1}{11}\left[64560x_{2n+4}-14436y_{2n+4}+132\right]$

xiii) $\frac{1}{22}\left[12y_{2n+3}-180y_{2n+2}+264\right]$

xiv) $\frac{1}{396}\left[12y_{2n+4}-3228y_{2n+2}+4752\right]$

xv) $\frac{1}{22}\left[180y_{2n+4}-3228y_{2n+3}+264\right]$

2.5 Each of the following expression represents a quintic integer

- $\frac{1}{(22)^3}\left[(64856x_{5n+5}-2904x_{5n+6})+5(134x_{n+1}-6x_{n+2})^3-5(64856x_{n+1}-2904x_{n+2})\right]$
- $\frac{1}{(396)^3}\left[(377299296x_{5n+5}-940896x_{5n+7})+5(2406x_{n+1}-6x_{n+3})^3-5(377299295x_{n+1}-940896x_{n+3})\right]$
- $\frac{1}{(11)^3}\left[(4840x_{5n+5}-726y_{5n+5})+5(40x_{n+1}-6y_{n+1})^3-5(4840x_{n+1}-729y_{n+1})\right]$
- $\frac{1}{(99)^3}\left[(5880600x_{5n+5}-58806y_{5n+6})+5(600x_{n+1}-6y_{n+2})^3-5(5880600x_{n+1}-58806y_{n+2})\right]$
- $\frac{1}{(1771)^3}\left[(33748105160x_{5n+5}-18818646y_{5n+7})+5(10760x_{n+1}-6y_{n+3})^3-5(33748105160x_{n+1}-18818646y_{n+3})\right]$
- $\frac{1}{(22)^3}\left[(1164504x_{5n+6}-64856x_{5n+7})+5(2406x_{n+2}-134x_{n+3})^3-5(1164504x_{n+2}-64856x_{n+3})\right]$
- $\frac{1}{(99)^3}\left[(392040x_{5n+6}-1313334y_{5n+5})+5(40x_{n+2}-134y_{n+1})^3-5(392040x_{n+2}-1313334y_{n+1})\right]$
- $\frac{1}{(11)^3}\left[(72600x_{5n+6}-16214y_{5n+6})+5(600x_{n+2}-134y_{n+2})^3-5(72600x_{n+2}-16214y_{n+2})\right]$
- $\frac{1}{(99)^3}\left[(105458760x_{5n+6}-1313334y_{5n+7})+5(10760x_{n+2}-134y_{n+3})^3-5(105458760x_{n+2}-1313334y_{n+3})\right]$
- $\frac{1}{(1771)^3}\left[(125457640x_{5n+7}-7546277046y_{5n+5})+5(40x_{n+3}-2406y_{n+1})^3-5(125457640x_{n+3}-7546277046y_{n+1})\right]$
- $\frac{1}{(99)^3}\left[(5880600x_{5n+7}-23581206y_{5n+6})+5(600x_{n+3}-2406y_{n+2})^3-5(5880600x_{n+3}-23581206y_{n+2})\right]$
- $\frac{1}{(11)^3}\left[(1301960x_{5n+7}-291126y_{5n+7})+5(10760x_{n+3}-2406y_{n+3})^3-5(1301960x_{n+3}-291126y_{n+3})\right]$
- $\frac{1}{(22)^3}\left[(968y_{5n+6}-14520y_{5n+5})+5(2y_{n+2}-30y_{n+1})^3-5(968y_{n+2}-14520y_{n+1})\right]$
- $\frac{1}{(396)^3}\left[(313632y_{5n+7}-84367008y_{5n+5})+5(2y_{n+3}-538y_{n+1})^3-5(313632y_{n+3}-84367008y_{n+1})\right]$
- $\frac{1}{(22)^3}\left[(14520y_{5n+7}-260392y_{5n+6})+5(30y_{n+3}-538y_{n+2})^3-5(14520y_{n+3}-260392y_{n+2})\right]$

2.6 Remarkable Observations

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in Table: 2 below.

Table 2: Hyperbolas

S. No	Hyperbolas	(X_n, Y_n)
1.	$20X_n^2 - Y_n^2 = 38720$	$[(134x_{n+1} - 6x_{n+2}), (40x_{n+2} - 600x_{n+1})]$
2.	$20X_n^2 - Y_n^2 = 12545280$	$[(2406x_{n+1} - 6x_{n+3}), (40x_{n+3} - 10760x_{n+1})]$
3.	$20X_n^2 - Y_n^2 = 9680$	$[(40x_{n+1} - 6y_{n+1}), (40y_{n+1} - 120x_{n+1})]$
4.	$20X_n^2 - Y_n^2 = 784080$	$[(600x_{n+1} - 6y_{n+2}), (40y_{n+2} - 2680x_{n+1})]$
5.	$20X_n^2 - Y_n^2 = 250915280$	$[(10760x_{n+1} - 6y_{n+3}), (40y_{n+3} - 48120x_{n+1})]$
6.	$20X_n^2 - Y_n^2 = 38720$	$[(2406x_{n+2} - 134x_{n+3}), (600x_{n+3} - 10760x_{n+2})]$
7.	$20X_n^2 - Y_n^2 = 784080$	$[(40x_{n+2} - 134y_{n+1}), (600y_{n+1} - 120x_{n+2})]$
8.	$20X_n^2 - Y_n^2 = 9680$	$[(600x_{n+2} - 134y_{n+2}), (600y_{n+2} - 2680x_{n+2})]$
9.	$20X_n^2 - Y_n^2 = 784080$	$[(10760x_{n+2} - 134y_{n+3}), (600y_{n+3} - 48120x_{n+2})]$
10.	$20X_n^2 - Y_n^2 = 250915280$	$[(40x_{n+3} - 2406y_{n+1}), (10760y_{n+1} - 120x_{n+3})]$
11.	$20X_n^2 - Y_n^2 = 784080$	$[(600x_{n+3} - 2406y_{n+2}), (10760y_{n+2} - 2680x_{n+3})]$
12.	$20X_n^2 - Y_n^2 = 9680$	$[(10760x_{n+3} - 2406y_{n+3}), (10760y_{n+3} - 48120x_{n+3})]$
13.	$20X_n^2 - Y_n^2 = 38720$	$[(2y_{n+2} - 30y_{n+1}), (134y_{n+1} - 6y_{n+2})]$
14.	$20X_n^2 - Y_n^2 = 12545280$	$[(2y_{n+3} - 538y_{n+1}), (2406y_{n+1} - 6y_{n+3})]$
15.	$20X_n^2 - Y_n^2 = 38720$	$[(30y_{n+3} - 538y_{n+2}), (2406y_{n+2} - 134y_{n+3})]$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in Table: 3 below

Table 3: Parabolas

S. No	Parabolas	(X_n, Y_n)
1.	$440X_n - Y_n^2 = 38720$	$[(134x_{2n+2} - 6x_{2n+3} + 44), (40x_{n+2} - 600x_{n+1})]$
2.	$7920X_n - Y_n^2 = 12545280$	$[(2406x_{2n+2} - 6x_{2n+4} + 792), (40x_{n+3} - 10760x_{n+1})]$
3.	$220X_n - Y_n^2 = 9680$	$[(40x_{2n+2} - 6y_{2n+2} + 22), (40y_{n+1} - 120x_{n+1})]$
4.	$1980X_n - Y_n^2 = 784080$	$[(600x_{2n+2} - 6y_{2n+3} + 198), (40y_{n+2} - 2680x_{n+1})]$
5.	$35420X_n - Y_n^2 = 250915280$	$[(10760x_{2n+2} - 6y_{2n+4} + 3542), (40y_{n+3} - 48120x_{n+1})]$
6.	$440X_n - Y_n^2 = 38720$	$[(2406x_{2n+3} - 134x_{2n+4} + 44), (600x_{n+3} - 10760x_{n+2})]$
7.	$1980X_n - Y_n^2 = 784080$	$[(40x_{2n+3} - 134y_{2n+2} + 198), (600y_{n+1} - 120x_{n+2})]$
8.	$220X_n - Y_n^2 = 9680$	$[(600x_{2n+3} - 134y_{2n+3} + 22), (600y_{n+2} - 2680x_{n+2})]$
9.	$1980X_n - Y_n^2 = 784080$	$[(10760x_{2n+3} - 134y_{2n+4} + 198), (600y_{n+3} - 48120x_{n+2})]$
10.	$35420X_n - Y_n^2 = 250915280$	$[(40x_{2n+4} - 2406y_{2n+2} + 3542), (10760y_{n+1} - 120x_{n+3})]$
11.	$1980X_n - Y_n^2 = 784080$	$[(600x_{2n+4} - 2406y_{2n+3} + 198), (10760y_{n+2} - 2680x_{n+3})]$
12.	$220X_n - Y_n^2 = 9680$	$[(10760x_{2n+4} - 2406y_{2n+4} + 22), (10760y_{n+3} - 48120x_{n+3})]$
13.	$440X_n - Y_n^2 = 38720$	$[(2y_{2n+3} - 30y_{2n+2} + 44), (134y_{n+1} - 6y_{n+2})]$
14.	$7920X_n - Y_n^2 = 12545280$	$[(2y_{2n+4} - 538y_{2n+2} + 792), (2406y_{n+1} - 6y_{n+3})]$
15.	$440X_n - Y_n^2 = 38720$	$[(30y_{2n+4} - 538y_{2n+3} + 44), (2406y_{n+2} - 134y_{n+3})]$

2.7 Generation of Pythagorean triangle

2.7.1 Let p , q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1} , \; q = y_{n+1}$$

Note that $p > q > 0$ treat p , q as the generators of Pythagorean triangle $T(X, Y, Z)$ where

$$X = 2pq , \; Y = p^2 - q^2 , \; Z = p^2 + q^2 , \; p > q > 0$$

Let A , P represents the area and perimeter of Pythagorean triangle. Then the following results are observed.

1. $X - 10Y + 9Z = 11$
2. $\frac{2A}{P} = x_{n+1} * y_{n+1}$
3. $X + Y - \frac{4A}{P}$ is written as the sum of two squares.
4. $3\left(X - \frac{4A}{P}\right)$ is a Nasty number.
5. $3(Z - Y)$ is a Nasty number.

2.7.2 Let p , q be the non-zero distinct integers such that

$$p = x_{n+1} + y_{n+1} , \; q = y_{n+1}$$

Note that $p > q > 0$ treat p , q as the generators of Pythagorean triangle $T(X, Y, Z)$ where

$$X = 2pq , \; Y = p^2 - q^2 , \; Z = p^2 + q^2 , \; p > q > 0$$

Let A , P represents the area and perimeter of Pythagorean triangle. In this case, the corresponding Pythagorean triangle satisfies the relation $Y - 40X + 39Z = 22$

3. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative pell equation $y^2 = 20x^2 - 11$. As the binary quadratic diophantine equations are rich in variety, one may search for the other choices of negative pell equations and determine their integer solutions along with suitable properties.

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